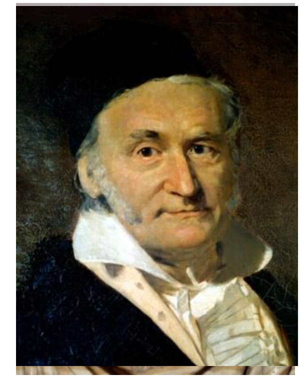
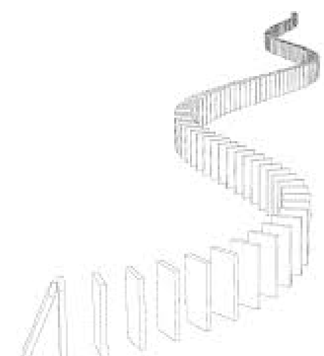
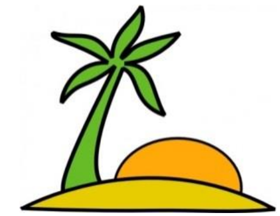


COMP 1002

Logic for Computer Scientists

Lecture 24

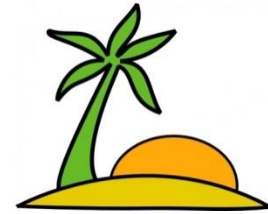




Puzzle

- A ship leaves a pair of rabbits on an island (with a lot of food).
- After a pair of rabbits reaches 2 months of age, they produce another pair of rabbits, and keep producing a pair every month thereafter.
- Which in turn starts reproducing every month when reaching 2 months of age…
- How many pairs of rabbits will be on the island in n months, assuming no rabbits die?

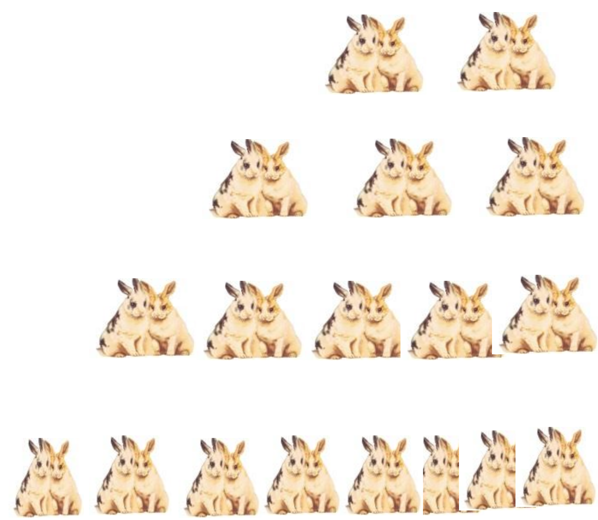
Fibonacci sequence



- A ship leaves a pair of rabbits on an island (with a lot of food).
- After a pair of rabbits reaches 2 months of age, they produce another pair of rabbits, and keep producing a pair every month thereafter. Which in turn starts reproducing every month when reaching 2 months of age…

- How many pairs of rabbits will be on the island in n months, assuming no rabbits die?

- Basis: $F_1 = 1,\ \mathrm{F}_2 = 1$
- Recurrence: $F_n = F_{n-1} + F_{n-2}$
- Sequence: 1,1,2,3,5,8,13…



Partial sums

- Properties of a sum ("linearity"):
  - $\sum_{i=m}^{n}(f(i)+g(i)) = \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i)$
  - $\sum_{i=m}^{n} c \cdot f(i) = c\sum_{i=m}^{n} f(i)$

Sum of arithmetic progression:

$s_n = c + nd$ for some c, $d \in \mathbb{R}$

Sequence: $c, c+d, c+2d, c+3d, \ldots, c+nd, \ldots$

Partial sum:

$$\sum_{i=0}^{n} s_n = \sum_{i=0}^{n}(c+id) = \sum_{i=0}^{n} c + \sum_{i=0}^{n} id =$$

$$= \mathrm{c(n+1)} + \mathrm{d}\sum_{i=0}^{n} i =$$

$$= c(n+1) + d\frac{n(n+1)}{2}$$

Sum of geometric progression:

$s_n = c \cdot r^n$ for some c, $r \in \mathbb{R}$

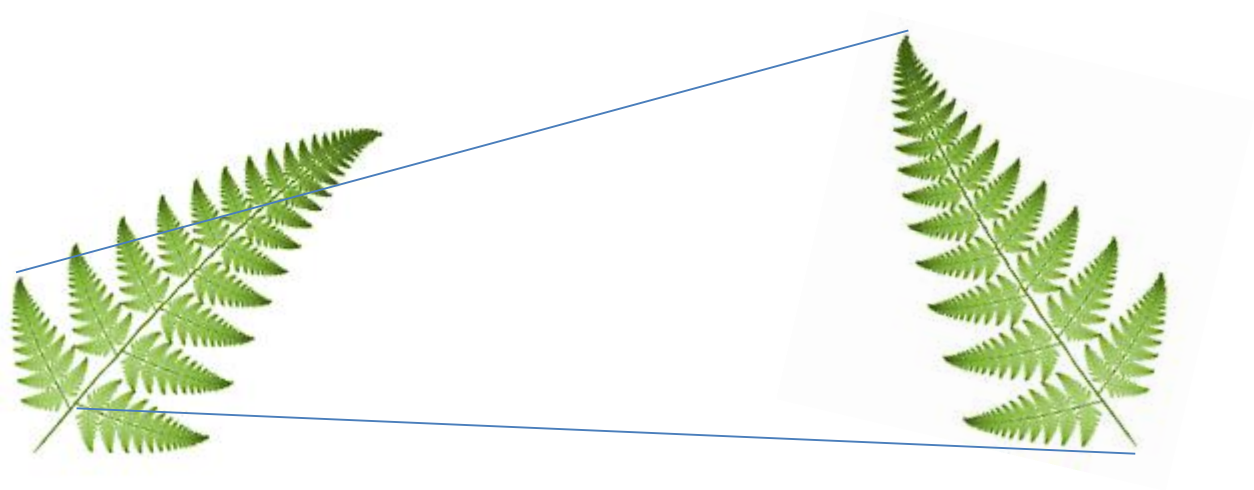
Sequence: $c, cr, cr^2, cr^3, \ldots, cr^n, \ldots$

Partial sum:

$$\sum_{i=0}^{n} s_n = \begin{cases} c(n+1), & if\ r = 1 \\ \frac{cr^{n+1}-c}{r-1}, & if\ r \neq 1 \end{cases}$$

Fractals

- Can use recursive definitions to define fractals
  - And draw them
  - And prove their properties.
- Self-similar: a part looks like the whole.



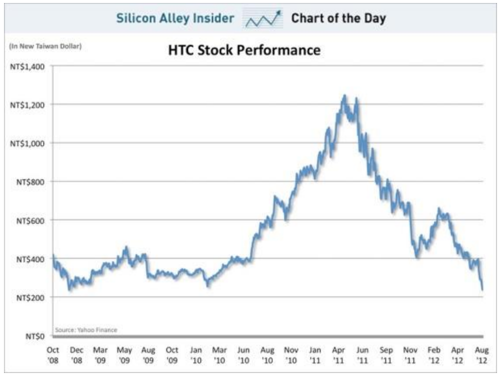
Fractals in nature

- A fern leaf

- Broccoli

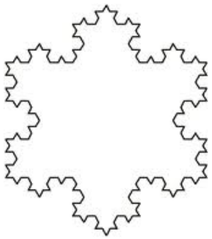
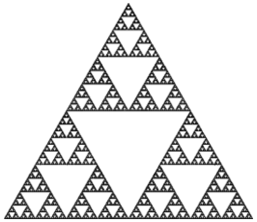
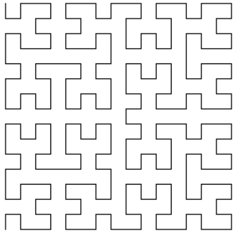
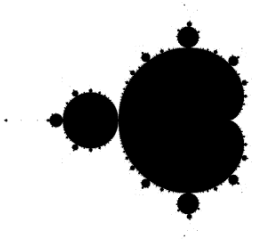
- Mountains

- Stock market



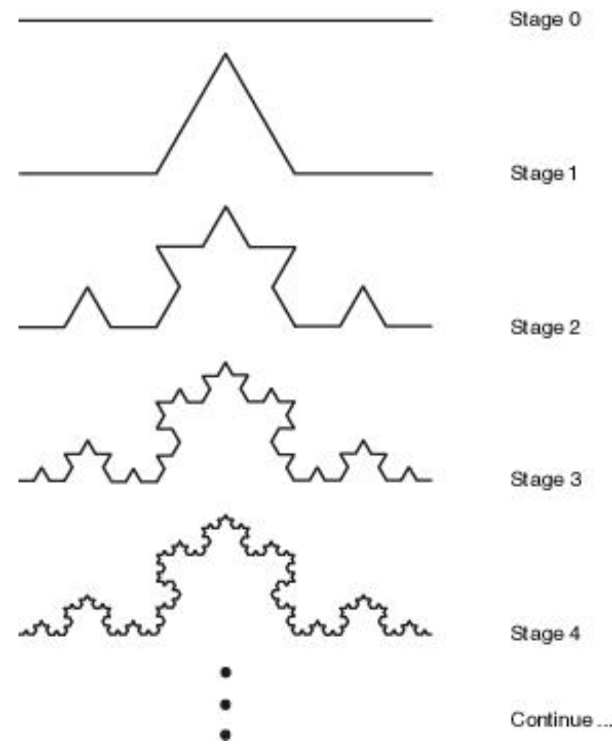
- Heat beat

Mathematical fractals

- Koch curve and snowflake 
- Sierpinski triangle, pyramid, carpet 
- Hilbert space-filling curve 
- Mandelbrot set 

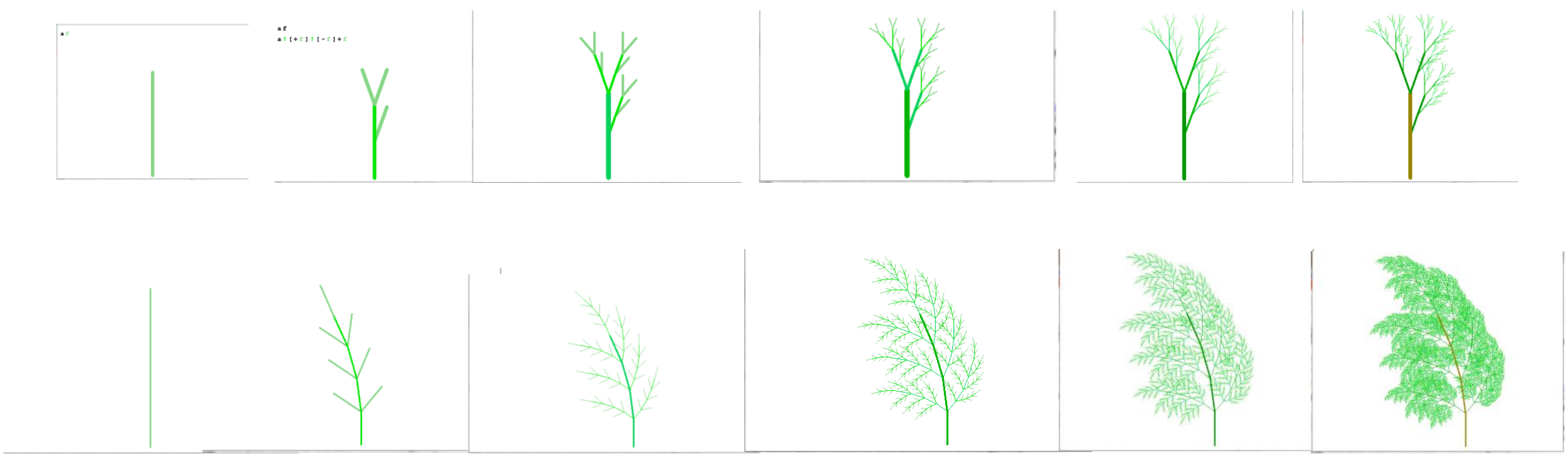
Koch curve

- *Basis:* an interval
- *Recursive step:* Replace the inner third of the interval with two of the same length
- …



Playing with fractals

- Fractal Grower by Joel Castellanos:
- http://www.cs.unm.edu/~joel/PaperFoldingFractal/paper.html



- So far, we talked about recursive definitions of sequences. We can also recursively define sets.
  - E.g: recursive definition of a set S={0,1}*
    - Basis: empty string is in S.
    - Recursive step: if $w \in S$, then $w0 \in S$ and $w1 \in S$
      - Here, w0 means string w with 0 appended at the end; same for w1
  - Alternatively:
    - Basis: empty string, 0 and 1 are in S.
    - Recursive step: if s and t are in S, then st $\in S$
      - here, st is concatenation: symbols of s followed by symbols of t
      - If s = 101 and t= 0011, then st = 1010011
  - Additionally, need a restriction condition: the set S contains only elements produced from basis using recursive step rule.

Recursive definitions of sets

- So far, we talked about recursive definitions of sequences. We can also recursively define sets.
  - E.g: recursive definition of a set S=$\{0,1\}^*$
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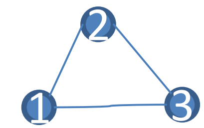
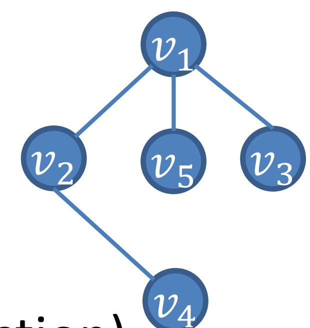
Trees

- In computer science, a **tree** is an undirected graph without cycles
  - **Undirected**: all edges go both ways, no arrows.
  - **Cycle**: sequence of edges going back to the same point.



Undirected cycle (not a tree)

- Recursive definition of trees:
  - Base: A single vertex v is a tree.
  - Recursion:
    - Let T be a tree, and v a new vertex.
    - Then a new tree consist of T, v, and an edge (connection) between some vertex of T and v.
  - Restriction:
    - Anything that cannot be constructed with this rule from this base is not a tree.



Arithmetic expressions

- Suppose you are writing a piece of code that takes an arithmetic expression and, say evaluates it.
  - "5*3-1", "40-(x+1)*7", etc
- How to describe a valid arithmetic expression? Define a set of all valid arithmetic expressions recursively.
  - Base: A number or a variable is a valid arithmetic expression.
    - 5, 100, x, a,
  - Recursion:
    - If A and B are valid arithmetic expressions, then so are (A), $A + B, A - B, A * B, A$ / B.
      - Constructing 40-(x+1)*7: first construct 40, x, 1, 7. Then x+1. Then (x+1). Then (x+1)*7, finally 40-(x+1)*7
      - Caveat: how do we know the order of evaluation? On that later.
  - Restriction: nothing else is a valid arithmetic expression.

Formulas

- What is a well-formed propositional logic formula?
  - $(p \vee \neg q) \wedge r \rightarrow (\neg p \rightarrow r)$
  - Base: a propositional variable p, q, r …
    - Or a constant $TRUE, FALSE$
  - Recursion:
    - If F and G are propositional formulas, so are (F), $\neg F$, $F \wedge G, F \vee G, F \rightarrow G, F \leftrightarrow G$.
  - And nothing else.

Formulas

- What is a well-formed predicate logic formula?
  - $\exists x \in D \; \forall y \in \mathbb{Z} \;\; P\big((x, y) \vee Q(x, z)\big) \wedge x = y$
  - Base: a predicate with free variables
    - P(x), x=y, …
  - Recursion:
    - If F and G are predicate logic formulas, so are (F), $\neg F$, $F \wedge G, F \vee G, F \rightarrow G, F \leftrightarrow G$.
    - If F is a predicate logic formula with a free variable x, then $\exists x \in D \; F$ and $\forall x \in D \; F$ are predicate logic formulas.
  - And nothing else.
    - So $\exists x \in People \;\; Likes(x, y \wedge x)$, $Likes(y \neq x)$ is not a well-formed predicate logic formula!

Grammars

- A general recursive definition for these is called a **grammar**.
  - In particular, here we have "context-free" grammars, where symbols have the same meaning wherever they are.

- A context-free grammar consists of
  - A set V of **variables** (using capital letters)
    - Including a **start variable** S.
  - A set Σ of **terminals** (disjoint from V; alphabet)
  - A set R of **rules**, where each rule consists of a variable from V and a string of variables and terminals.
    - If $A \rightarrow w$ is a rule, we say variable A **yields** string w.
      - This is not the same "→ " as implication, a different use of the same symbol.
    - We use shortcut "|" when the same variable might yield several possible strings: $A \rightarrow w_1 | \, w_2 | \dots |w_k$
    - Can use A again within the rule: Recursion!
      - Different occurrences of the same variable can be interpreted as different strings.
    - When left with just terminals, a string is **derived**.

Grammars

- **A language generated by a grammar** consists of all strings of terminals that can be derived from the start variable by applying the rules.
  - All strings are derived by repeatedly applying the grammar rules to each variable until there are no variables left (just the terminals).
  - Language {1, 00} consisting of two strings 1 and 00
    - $S \rightarrow 1 \mid 00$
      - Variables: S. Terminals: 1 and 00.
  - Language of all strings over {0,1} with all 0s before all 1s.
    - $S \rightarrow 0S \mid S1 \mid \_$
      - Variables: S. Terminals: 0 and 1.

More context-free grammars

- Propositional formulas.
  1. $F \rightarrow \; F \vee F$
  2. $\mathrm{F} \rightarrow F \wedge F$
  3. $F \rightarrow \; \neg F$
  4. $F \rightarrow \; (F)$
  5. $F \rightarrow p \mid q \mid r \mid TRUE \mid FALSE$
     - Here, the only variable is F (it is a start variable), and terminals are ∨,∧ ,¬, (,), $p, q, r, TRUE, FALSE$
     - To obtain $(p \vee \neg q) \wedge r$, first apply rule 2, then rule 4 to strip parentheses from $(p \vee \neg q)$, then rule 1, then rule 5 to get p, then rule 3, then rule 5 to get q, then rule 5 to get r.

- Arithmetic expressions.
  - $EXPR \rightarrow EXPR + EXPR \mid \; EXPR \; - EXPR \mid EXPR \; * EXPR \mid EXPR \; / \; EXPR \mid (EXPR) \; \mid NUMBER \mid$-NUMBER
  - NUMBER $\rightarrow 0DIGITS \mid \ldots \mid 9DIGITS$
  - $DIGITS \; \rightarrow$ _$\mid NUMBER$
    - Here, _ stands for empty string. Variables: EXPR, NUMBER, DIGITS (S is starting). Terminals: +,-,*, /, 0,…,9,(,).
    - We used separate NUMBER to avoid multiple “-”.
    - And separate DIGITS to have an empty string to finish writing a number, but to avoid an empty number.

Encoding order of precedence

- Easier to specify in which order to process parts of the formula.
  - Better grammar for arithmetic expressions (for simplicity, with x,y,z instead of numbers):
    1. $EXPR \rightarrow EXPR + TERM \mid EXPR - TERM \mid TERM$
    2. $TERM \rightarrow TERM * FACTOR \mid TERM \,/\, FACTOR \mid FACTOR$
    3. $FACTOR \rightarrow (EXPR)$ | x | y | z
  - Here, variables are EXPR, TERM and FACTOR (with EXPR a starting variable).
  - Now can encode precedence.
    - And put parentheses more sensibly.

Parse trees.

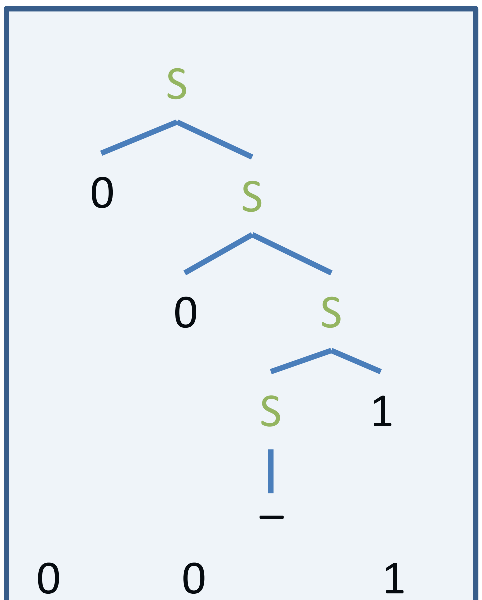
- Visualization of derivations: parse trees.

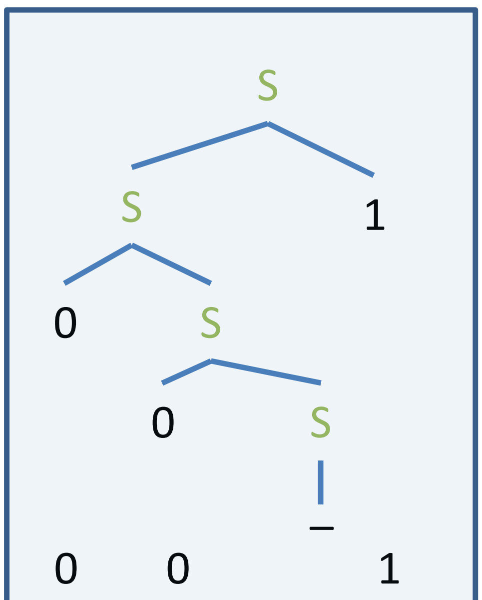
  1. $EXPR \to EXPR + TERM \mid EXPR - TERM \mid TERM$
  2. $TERM \to TERM * FACTOR \mid TERM \,/\, FACTOR \mid FACTOR$
  3. $FACTOR \to (EXPR) \mid x \mid y \mid z$

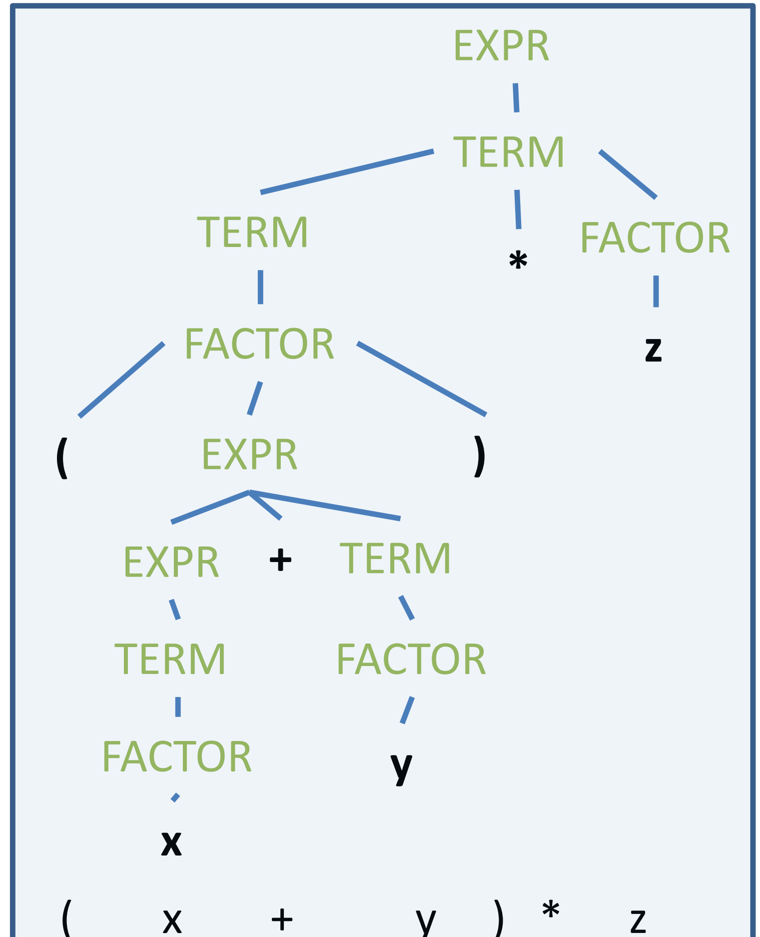
  - String (x+y)*z

  – Simpler example:

    - $S \to 0S \mid S1 \mid \_$
    - String 001







Puzzle

- Do the following two English sentences have the same parse trees?
  - Time flies like an arrow.

  - Fruit flies like an apple.