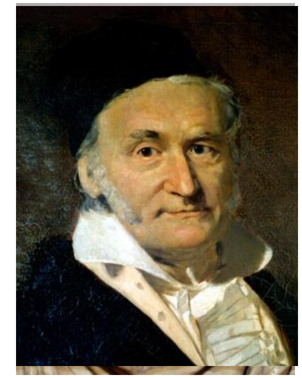
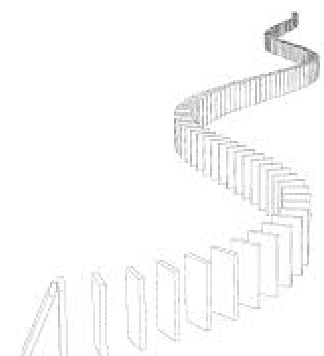


COMP 1002

Logic for Computer Scientists

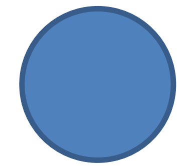
Lecture 23

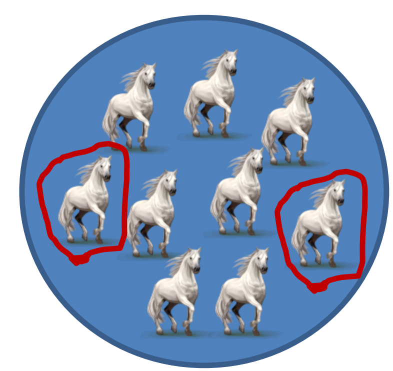
B 5 2 J



Puzzle: all horses are white

- Claim: all horses are white.
- Proof (by induction):
  - P(n): any n horses are white.
  - Base case: P(0) holds vacuously
  - Induction hypothesis: any k horses are white.
  - Induction step: if any k horses are white, then any k+1 horses are white.
    - Take an arbitrary set of k+1 horses. Take a horse out.
      - The remaining k horses are white by induction hypothesis.
    - Now put that horse back in, and take out another horse.
      - Remaining k horses are again white by induction hypothesis.
    - Therefore, all the k+1 horses in that set are white.
  - By induction, all horses are white.





Strong induction: product of primes

- Strong Induction principle (general form):
  - $\left(\exists b \geq a \ \ \forall c \in \{a..b\} \mathrm{P(c)}\right) \land \forall\, k > b \ \ (\forall\, i \ \in \{a, \dots, k-1\} \ \mathrm{P(i)}) \to \mathrm{P(k)})$
    $\to \forall x \in \mathbb{N} \left(x \geq a \to \ P(x)\right)$
- Theorem: Every natural number ≥ 2 is a product of powers of prime numbers.
  - $\boldsymbol{P(n)}$: $\exists\, m \in \mathbb{N} \ \exists\, p_1, \dots, p_m, \ d_1, \dots, d_m$ such that $n = p_1^{d_1} \cdot p_2^{d_2} \cdot \dots \ \cdot p_m^{d_m}$
    - For example, $24 = 2^3 \cdot 3^1$, $23 = 23^1$
  - **Base case:** $a = b = 2$. 2 is a prime, so P(2) holds with $m = 1, p_1 = 2, d_1 = 1$.
  - **Induction hypothesis**: Let k be an arbitrary integer > 2. Assume that

$\forall i \in \{2, \dots, k-1\} \,\exists\, m \in \mathbb{N} \ \exists\, p_1, \dots, p_m, \ d_1, \dots, d_m \ \ i = p_1^{d_1} \cdot p_2^{d_2} \cdot \dots \ \cdot p_m^{d_m}$

  - **Induction step**. Show that the induction hypothesis implies that

$$\exists\, m' \in \mathbb{N} \ \exists\, q_1, \dots, q_{m'}, d'_1, \dots, d'_{m'} \ \ k = q_1^{d'_1} \cdot q_2^{d'_2} \cdot \dots \ \cdot q_{m'}^{d'_{m'}}$$

Strong induction: product of primes

- **Induction hypothesis**: Let k be an arbitrary integer > 2. Assume that

$\forall i \in \{2, \ldots, k-1\} \; \exists\, m \in \mathbb{N} \;\; \exists\, p_1, \ldots, p_m, \; d_1, \ldots, d_m \;\; i = p_1^{d_1} \cdot p_2^{d_2} \cdot \ldots \; \cdot p_m^{d_m}$

- **Induction step**. Show that the induction hypothesis implies that

$$\exists\, m' \in \mathbb{N} \;\; \exists\, q_1, \ldots, q_{m'}, d'_1, \ldots, d'_{m'} \;\; k = q_1^{d'_1} \cdot q_2^{d'_2} \cdot \ldots \; \cdot q_{m'}^{d'_{m'}}$$

  - Rename variables to distinguish from induction hypothesis.

- Case 1: k is prime. Then $m' = 1, \; q_1 = k, \; d'_1 = 1$
- Case 2: k is not prime. Then $k = k_1 \cdot k_2$. By induction hypothesis, both k_1 and k_2 are products of primes.
  - Here is where *strong* induction is useful.
  - Then get $m', \; q_1 \ldots q_{m'}, d'_1, \ldots, d'_{m'}$ as follows:
    - Let m' be the number of unique primes in k_1, k_2 (that is, if a prime occurs in both, count it once.) Then $q_1 \ldots q_{m'}$ is the set of unique primes in k_1 and k_2.
    - If a prime p occurs in both k_1 and k_2, it will be in k with power which is sum of its powers in k_1, k_2
    - If it occurs in only one of k_1, k_2, put it in k with the same power it had where it appeared.

- By strong induction, every number ≥ 2 is a product of prime powers.

Recurrences and sequences

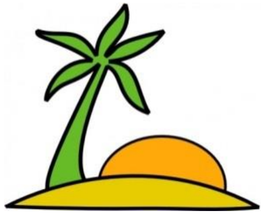
- To define a sequence (of things), describe a process generating it.
  - Sequence: enumeration of objects $s_1, s_2, s_3, \ldots, s_n, \ldots,$
    - Sometimes use notation $\{s_n\}$ for the sequence (set of elements forming a sequence)
  - **Basis (initial conditions):** what are the first (few) element(s) in the sequence.
    - $\sum_{i=m}^{m} i$ = m.
    - 0! = 1.   1!=1.
    - $\mathrm{A}_0 = \emptyset$
  - **Recurrence (recursion step, inductive definition)**: a rule to make a next element from already constructed ones.
    - $\sum_{i=m}^{n+1} i = (\sum_{i=m}^{n} i) + (n+1)$. Here, assume that $m \leq n$, make it 0 otherwise
    - (n+1)! = n! · (n+1)
    - $A_{n+1} = \mathcal{P}(A_n)$
- Resulting sequences:
  - m, 2m+1, 3m+3, …
  - 1, 2,6, 24, 120, …
  - $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \left\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\right\}, \ldots$

Special sequences

- Arithmetic progression:
  - Sequence: $c, c+d, c+2d, c+3d, \ldots, c+nd, \ldots$
  - Recursive definition:
    - Basis: $s_0 = \mathrm{c}$, for some c$\in \mathbb{R}$
    - Recurrence: $s_{n+1} = \mathrm{s_n} + \mathrm{d}$, where $d \in \mathbb{R}$ is a fixed number.

- Geometric progression:
  - Sequence: $c, cr, cr^2, cr^3, \ldots, cr^n, \ldots$
  - Recursive definition:
    - Basis: $s_0 = \mathrm{c}$, for some c$\in \mathbb{R}$
    - Recurrence: $s_{n+1} = \mathrm{s_n} \cdot r$, where r$\in \mathbb{R}$ is a fixed number.

Puzzle



- A ship leaves a pair of rabbits on an island (with a lot of food).
- After a pair of rabbits reaches 2 months of age, they produce another pair of rabbits, and keep producing a pair every month thereafter.
- Which in turn starts reproducing every month when reaching 2 months of age…
- How many pairs of rabbits will be on the island in n months, assuming no rabbits die?