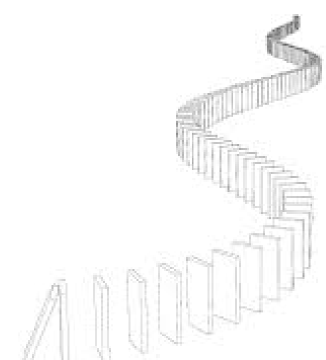
COMP 1002

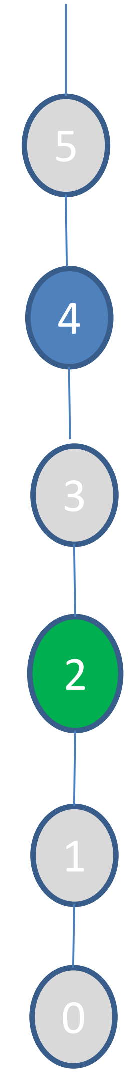
Logic for Computer Scientists

Lecture 22



Well-ordering principle

- **Theorem:** Any non-empty subset of natural numbers contains the least element
  - With respect to the usual total order $x \leq y$
    - There is smallest positive even number. Smallest composite number. Smallest square…
  - In general, if there is a property which is not true for some natural numbers, there is a smallest natural number for which it is not true.
  - Very useful for proofs!
    - We saw it in the proof that there are infinitely many primes



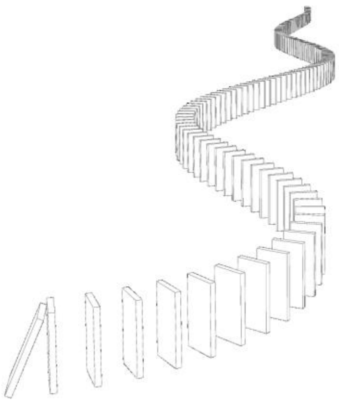
Sum of numbers formula

Gauss' proof:
1 + 2 + … + 99 + 100 +
100 + 99 + … + 2 + 1 =
101 + 101+ … +101 + 101 =100*101
So 1+2+ … + 99 + 100 =$\frac{100*101}{2}$
Works for any n, not just n=100

- Claim: for any n∈ $\mathbb{N}$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$
- Proof.
  - Suppose not.
  - Let S be a set of all numbers n' such that $\sum_{i=0}^{n'} i \neq \frac{n'(n'+1)}{2}$. By well-ordering principle, if $S \neq \emptyset$, then there is the least number k in S.
  - Case 1: k=0. But $\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$. So formula works for k=0.
  - Case 2: k>0. Then $k - 1 \geq 0$.
    - So $\sum_{i=0}^{k} i = (\sum_{i=0}^{k-1} i)$ +k.
    - As k is the smallest bad number, the formula works for k-1.
    - So $\sum_{i=0}^{k-1} i = \frac{(\mathrm{k}-1)\mathrm{k}}{2}$
    - Now, $\sum_{i=0}^{k} i = (\sum_{i=0}^{k-1} i)$ +k = $\frac{(\mathrm{k}-1)\mathrm{k}}{2} + \mathrm{k} = \frac{\mathrm{k}^2-\mathrm{k}+2\mathrm{k}}{2} = \frac{\mathrm{k}^2+k}{2} = \frac{k(k+1)}{2}$
    - So the formula works for k>0, too.
  - Contradiction. So S is empty, thus the formula works for all $n \in \mathbb{N}$.

Mathematical induction

- Want to prove a statement $\forall x \in \mathbb{N}\ \ P(x)$.
  - Check that $P(0)$ holds
  - And whenever $P(k)$ does not hold for some k, $P(k-1)$ does not hold either
    - Contradicting well-ordering principle.
    - Contrapositive:
      - if P(k-1) holds for arbitrary k,
      - then P(k) also must be true.
  - Conclude that $\forall x \in \mathbb{N}\ \ P(x)$



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Proving that P(0) holds
is called the **base case**.

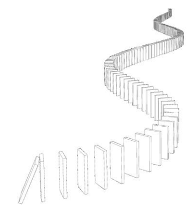
That P(k-1) holds is an **induction hypothesis**

Proving that P(k-1) → P(k)
Is the **induction step**

Mathematical Induction principle:
If $P(0) \wedge \forall k \in \mathbb{N}\ \ P(k) \rightarrow P(k+1)$ then $\forall x \in \mathbb{N}\ P(x)$

Sum of numbers by induction

- Claim: for any n∈ ℕ, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$
- Proof (by induction).
  - P(n) is $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ (*statement* we are proving by induction on n)
  - *Base case:* n=0. Then $\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$.
  - *Induction hypothesis:* Assume that $\sum_{i=0}^{k-1} i = \frac{(k-1)k}{2}$ for an arbitrary k >0
    - That is, for an arbitrary number n=k-1 ∈ ℕ
    - Can take k instead of k-1, but k-1 makes calculations simpler.
  - *Induction step:* show that P(k-1) implies P(k).
    - $\sum_{i=0}^{k} i = (\sum_{i=1}^{k-1} i)$ +k.
    - By induction hypothesis, $\sum_{i=1}^{k-1} i = \frac{(\mathrm{k}-1)\mathrm{k}}{2}$
    - Now, $\sum_{i=1}^{k} i = (\sum_{i=1}^{k-1} i)$ +k = $\frac{(\mathrm{k}-1)\mathrm{k}}{2} + \mathrm{k} = \frac{\mathrm{k}^2-\mathrm{k}+2\mathrm{k}}{2} = \frac{\mathrm{k}^2+k}{2} = \frac{k(k+1)}{2}$
  - *By induction*, therefore, P(n) holds for all $n \in \mathbb{N}$.



Changing the base case

- Mathematical Induction principle:
  - $(\mathrm{P}(0) \wedge \forall\, k \in \mathbb{N}\ \ \mathrm{P(k)} \rightarrow \mathrm{P(k+1)}) \ \rightarrow \forall x \in \mathbb{N}\, P(x)$

- What if want to prove it only for $x \geq a$?
  - Make a the base case (when $a \geq 0$). For the rest, assume $k \geq a$.
  - $(\mathrm{P(a)} \wedge \forall\, k \geq a\ \ \mathrm{P(k)} \rightarrow \mathrm{P(k+1)}) \ \rightarrow \forall x \geq a\ \ P(x)$
    - Here, $\forall x \geq a\ \ P(x)$ is a shorthand for $\forall x \in \mathbb{N}\ \ \left(x \geq a \rightarrow P(x)\right)$
  - To prove it works, prove P(n’) where n’=n-a.

Changing the base case

Example: show that for all $n \geq 4,\ 2^n \geq n^2$

- Predicate $P(n)$: $2^n \geq n^2$
- Base case: n=4. $2^4 = 16 = 4^2$
- Induction hypothesis: assume that for an arbitrary $k \geq a$, $2^k \geq k^2$
- Induction step: show that $2^k \geq k^2$ implies $2^{k+1} \geq (k+1)^2$
  - $2^{k+1} = 2 \cdot 2^k = 2^k + 2^k \geq k^2 + k^2$
  - $(k+1)^2 = k^2 + 2k + 1.$
  - Want: $k^2 + k^2 \geq k^2 + 2k + 1$, so $k^2 \geq 2k + 1$
    - Dividing both sides of the inequality by k: $k \geq 2 + \frac{1}{k}$
    - Since k ≥ 4, and $2 + \frac{1}{k} \leq 3$, $2 + \frac{1}{k} \leq 3 < 4 \leq k$. So $k \geq 2 + \frac{1}{k}$ and thus $k^2 \geq 2k + 1$
  - So $2^{k+1} = 2 \cdot 2^k = 2^k + 2^k \geq k^2 + k^2 \geq k^2 + 2k + 1 = (k+1)^2$
- By induction, for all $n \geq 4,\ 2^n \geq n^2$

- Corollary: as n grows, an algorithm running in time n^2 will quickly outperform an algorithm running in time 2^n

Coins by induction

- Coins: $\forall x \in \mathbb{N}$, if x >7 then $\exists\ y, z \in \mathbb{N}$ such that x = 3y+5z. So any amount >7 can be paid with 3s and 5s.
  - Predicate $P(n)$: $\exists y, z \in \mathbb{N}\ \ n = 3y + 5z$
  - *Base case:* $P(8)$: $y = 1, z = 1, 8 = 3 \cdot 1 + 5 \cdot 1$
  - Induction hypothesis: suppose that $P(k)$ holds
    - That is, $\exists y, z$ such that $k = 3y + 5z$
  - Induction step: show that $P(k+1)$ holds
    - That is, show that $\exists y', z' \in \mathbb{N}\ \ such\ that\ \ \mathrm{k} + 1 = 3\mathrm{y}' + 5\mathrm{z}'$
    - Construct y', z' from y, z
    - If $z > 0, then\ y' = y + 2, z' = z - 1$
    - If $z = 0$, then $y' = y - 3$, z'=2
  - Therefore, for every $x \in \mathbb{N}$, if x >7 then x=3y+5z for some $y, z \in \mathbb{N}$.

Strong induction

- For our coins problem, needed not just P(k-1), but P(k-3), and to look at three cases.
- Mathematical Induction principle:
  - $(P(0) \land \forall\, k \in \mathbb{N}\ \ P(k) \to P(k+1)) \ \to \forall x \in \mathbb{N}\ P(x)$
- Strong Induction principle:
  - $\Big(\exists b \in \mathbb{N}\ \ \forall c \in \mathbb{N}\,\big(0 \le c \land c \le b \to\ P(c)\big)\Big)$

    $\land \forall\, k > b\ \ (\forall\, i\ \in \{0, \dots, k-1\}\ P(i)) \to\ P(k))$

    $\to \forall x \in \mathbb{N}\ P(x)$

Equivalence of variants of induction

- Strong induction seems stronger… but in fact, *mathematical induction, strong induction and well-order principles are equivalent to each other.*
  - So choose the most convenient one.

- Can prove induction from well-ordering principle
  - Look at the smallest k such that $P(k)$ does not hold
- Can prove strong induction statement by normal induction.
  - Prove $P'(n) = \forall i < n\ P(n)$ by induction.
- Can prove well-ordering principle from strong induction.

Puzzle: coins

- A not-too-far-away country recently got rid of a penny coin, and now everything needs to be rounded to the nearest multiple of 5 cents…
  - Suppose that instead of just dropping the penny, they would introduce a 3 cent coin.
    - Like British three pence.
  - What is the largest amount that cannot be paid by using only existing coins (5, 10, 25) and a 3c coin?

7c
Any number n >7 can be paid with 3,5,10,25 coins (even just 3 and 5).

Well-ordering principle

- Any non-empty subset of natural numbers contains the least element
  - With respect to the usual total order $x \leq y$
  - Very useful for proofs!
- Coins: $\forall x \in \mathbb{N}$, if x >7 then $\exists\ y, z \in \mathbb{N}$ such that x = 3y+5z. So any amount >7 can be paid with 3s and 5s.
  - Suppose, for the sake of contradiction, that there are amounts greater than 7 which cannot be paid with 3s and 5s.
  - *Take a set S of all such amounts. Since $S \subseteq \mathbb{N}$, and we assumed that $S \neq \emptyset$, by well-ordering principle S has the least element. Call it n.*
  - Now, look at n-3; it cannot be paid by 3s and 5s either.
  - Since n is the least element of S, $n - 3 \leq 7 < n$
  - 3 cases:
    - n-3 = 7. Then n=10=2*5.
    - n-3 = 6. Then n=9=3*3
    - n-3 = 5. Then n=8=3+5.
  - In all three cases, got a contradiction.
  - Therefore, for every $x \in \mathbb{N}$, if x >7 then x=3y+5z for some $y, z \in \mathbb{N}$.

Strong induction

- Strong Induction principle (general form):
  - $(\exists b \in \mathbb{N}\ \forall c \in \mathbb{N}\ (a \leq c \wedge c \leq b \rightarrow \ \mathrm{P(c)})$
    $\wedge \forall\ k > b\ \ (\forall\ i\ \ \in \{a, \ldots, k-1\}\ \ \mathrm{P(i)}) \rightarrow \mathrm{P(k)})$
    $\rightarrow \forall x \in \mathbb{N}\ (x \geq a \rightarrow \ P(x))$
- Coins: $\forall x \in \mathbb{N}$, if x >7 then $\exists\ y, z \in \mathbb{N}$ such that x = 3y+5z.
  - P(n): $\exists\ y, z \in \mathbb{N}\ \ n = 3y + 5z$. Also, a=8.
  - Base cases: b = 10, so $c \in \{8,9,10\}$
    - n=8. $8 = 3 \cdot 1 + 5 \cdot 1$, so y=1, z=1.
    - n=9. 9=3$\cdot$ 3, y=3, z=0
    - n=10. 10=5 $\cdot$ 5. y=0, z=2.
  - Induction hypothesis: Let k be an arbitrary integer such that $k > 10$. Assume that for all $i \in \mathbb{N}$ such that $8 \leq i\ < k\ \exists\ y_i, z_i \in \mathbb{N}\ \ \ i = 3y_i + 5z_i$
  - Induction step. Show that induction hypothesis implies that $\exists\ y, z \in \mathbb{N}\ \ k = 3y + 5z$
    - Since $k \geq b,\ \ k - 3 \geq a$. So by induction hypothesis $\exists\ y_{k-3}, z_{k-3} \in \mathbb{N}\ \ \ k - 3 = 3y_{k-3} + 5z_{k-3}$. Now take z=$z_{k-3}$ and y = y_{k-3} +1. Then k = 3y+5z.
  - By strong induction, get that for all x > 7, $\exists\ y, z \in \mathbb{N}$ such that x = 3y+5z.

Puzzle: all horses are white

- Claim: all horses are white.
- Proof (by induction):
  - P(n): any n horses are white.
  - Base case: P(0) holds vacuously
  - Induction hypothesis: any k horses are white.
  - Induction step: if any k horses are white, then any k+1 horses are white.
    - Take an arbitrary set of k+1 horses. Take a horse out.
      - The remaining k horses are white by induction hypothesis.
    - Now put that horse back in, and take out another horse.
      - Remaining k horses are again white by induction hypothesis.
    - Therefore, all the k+1 horses in that set are white.
  - By induction, all horses are white.