COMP 1002

Logic for Computer Scientists

Lecture 21
Admin stuff

• Assignment 4 is posted
  – Due Tuesday, March 12
Partial and total orders

• A binary relation $R \subseteq A \times A$ is an order if $R$ is
  – Reflexive $\forall x \in A \ R(x, x)$
  – Anti-symmetric $\forall x, y \in A \ R(x, y) \land R(y, x) \rightarrow x = y$
  – Transitive $\forall x, y, z \in A \ R(x, y) \land R(y, z) \rightarrow R(x, z)$

• $R$ is a total order if $\forall x, y \in A \ R(x, y) \lor R(y, x)$
  • That is, every two elements of $A$ are related.
  • E.g. $R_1 = \{(x, y)|x, y \in \mathbb{Z} \land x \leq y\}$ is a total order.
  • So is alphabetical order of English words.
  • But not $R_2 = \{(x, y)|x, y \in \mathbb{Z} \land x < y\}$
    – not reflexive, so not an order.

• Otherwise, $R$ is a partial order.
  • $\text{SUBSETS} = \{(A, B) | A, B \text{ are sets } \land A \subseteq B \}$ is a partial order.
    – Reflexive: $\forall A, \ A \subseteq A$
    – Anti-symmetric: $\forall A, B \ A \subseteq B \land B \subseteq A \rightarrow A = B$
    – Transitive: $\forall A, B, C \ A \subseteq B \land B \subseteq C \rightarrow A \subseteq C$
    – Not total: if $A = \{1,2\}$ and $B = \{1,3\}$, then neither $A \subseteq B$ nor $B \subseteq A$
  • $\text{DIVISORS} = \{(x,y)| x, y \in \mathbb{N} \land x, y \geq 2 \land \exists z \in \mathbb{N} \ y = z \cdot x\}$ is a partial order.
Partial and total orders

• A binary relation \( R \subseteq A \times A \) is an order if \( R \) is
  – Reflexive, Anti-symmetric, Transitive
  – \( R \) is a total order if \( \forall x, y \in A \ R(x, y) \vee R(y, x) \)
    • \( R_1 = \{(x, y) | x, y \in \mathbb{Z} \land x \leq y\} \) is a total order.
    • So is alphabetical order of English words.
  – Otherwise, \( R \) is a partial order.
    • \( SUBSETS = \{(A, B) \mid A, B \text{ are sets} \land A \subseteq B \} \)
    • \( DIVISORS = \{(x, y) \mid x, y \in \mathbb{N} \land x, y \geq 2 \land \exists z \in \mathbb{N} \ y = z \cdot x\} \)
• An order may have minimal and maximal elements (maybe multiple)
  – \( x \in A \) is minimal in \( R \) if \( \forall y \in A \ y \neq x \rightarrow \neg R(y, x) \)
    and maximal if \( \forall y \in A \ y \neq x \rightarrow \neg R(x, y) \)
  – \( \emptyset \) is minimal in \( SUBSETS \) (its unique minimum);
    universe is maximal (its unique maximum).
  – All primes are minimal in \( DIVISORS \), and there are no maximal elements.
Well-ordering principle

- **Theorem**: Any non-empty subset of natural numbers contains the least element
  - With respect to the usual total order $x \leq y$
    - There is smallest positive even number. Smallest composite number. Smallest square...
  - In general, if there is a property which is not true for some natural numbers, there is a smallest natural number for which it is not true.
  - Very useful for proofs!
    - We saw it in the proof that there are infinitely many primes
Puzzle: coins

- A not-too-far-away country recently got rid of a penny coin, and now everything needs to be rounded to the nearest multiple of 5 cents...
  - Suppose that instead of just dropping the penny, they would introduce a 3 cent coin.
    - Like British three pence.
  - What is the largest amount that cannot be paid by using only existing coins (5, 10, 25) and a 3c coin?

7c
Any number n > 7 can be paid with 3, 5, 10, 25 coins (even just 3 and 5).
Well-ordering principle

- Any non-empty subset of natural numbers contains the least element
  - With respect to the usual total order $x \leq y$
  - Very useful for proofs!
- Coins: $\forall x \in \mathbb{N},$ if $x > 7$ then $\exists y, z \in \mathbb{N}$ such that $x = 3y + 5z$. So any amount $> 7$ can be paid with 3s and 5s.
  - Suppose, for the sake of contradiction, that there are amounts greater than 7 which cannot be paid with 3s and 5s.
  - Take a set $S$ of all such amounts. Since $S \subseteq \mathbb{N},$ and we assumed that $S \neq \emptyset,$ by well-ordering principle $S$ has the least element. Call it $n$.
  - Now, look at $n - 3$; it cannot be paid by 3s and 5s either.
  - Since $n$ is the least element of $S,$ $n - 3 \leq 7 < n$
  - 3 cases:
    - $n - 3 = 7.$ Then $n = 10 = 2 \times 5.$
    - $n - 3 = 6.$ Then $n = 9 = 3 \times 3$
    - $n - 3 = 5.$ Then $n = 8 = 3 + 5.$
  - In all three cases, got a contradiction.
  - Therefore, for every $x \in \mathbb{N},$ if $x > 7$ then $x = 3y + 5z$ for some $y, z \in \mathbb{N}.$
Sums, products and sequences

• How to write long sums, e.g., 1+2+... (n-1)+n concisely?
  – Sum notation (“sum from 1 to n”): \( \sum_{i=1}^{n} i = 1 + 2 + \ldots + n \)
    • If n=3, \( \sum_{i=1}^{3} i = 1+2+3=6. \)
    • The name “i“ does not matter. Could use another letter not yet in use.

• In general, let \( f: \mathbb{Z} \rightarrow \mathbb{R}, \ m, n \in \mathbb{Z}, m \leq n. \)
  – \( \sum_{i=m}^{n} f(i) = f(m) + f(m+1) + \ldots + f(n) \)
    • If m=n, \( \sum_{i=m}^{n} f(i) = f(m)=f(n). \)
    • If n=m+1, \( \sum_{i=m}^{n} f(i) = f(m)+f(m+1) \)
    • If n>m, \( \sum_{i=m}^{n} f(i) = (\sum_{i=m}^{n-1} f(i)) + f(n) \)
    • Example: \( f(x) = x^2. \) \( 2^2 + 3^2 + 4^2 = \sum_{i=2}^{4} i^2 = 29 \)

• Similarly for product notation (product from m to n):
  – \( \prod_{i=m}^{n} f(i) = f(m) \cdot f(m+1) \cdot \ldots \cdot f(n) = (\prod_{i=m}^{n-1} f(i)) \cdot f(n) \)
  – For \( f(x) = x, \) \( 2 \cdot 3 \cdot 4 = \prod_{i=2}^{4} i = 24 \)
  – \( 1 \cdot 2 \cdot \ldots \cdot n = \prod_{i=1}^{n} i = n! \) (n factorial)
Sum of numbers formula

- Claim: for any $n \in \mathbb{N}$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

- Proof.
  - Suppose not.
  - Let $S$ be a set of all numbers $n'$ such that $\sum_{i=0}^{n'} i \neq \frac{n'(n'+1)}{2}$. By well-ordering principle, if $S \neq \emptyset$, then there is the least number $k$ in $S$.
  - Case 1: $k=0$. But $\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$. So formula works for $k=0$.
  - Case 2: $k>0$. Then $k - 1 \geq 0$.
    - So $\sum_{i=0}^{k-1} i = (\sum_{i=0}^{k-1} i) + k$.
    - As $k$ is the smallest bad number, the formula works for $k-1$.
    - So $\sum_{i=0}^{k-1} i = \frac{(k-1)k}{2}$
    - Now, $\sum_{i=0}^{k} i = (\sum_{i=0}^{k-1} i) + k = \frac{(k-1)k}{2} + k = \frac{k^2-k+2k}{2} = \frac{k^2+k}{2} = \frac{k(k+1)}{2}$
    - So the formula works for $k>0$, too.
  - Contradiction. So $S$ is empty, thus the formula works for all $n \in \mathbb{N}$.

Gauss' proof:

1 + 2 + ... + 99 + 100 + 100 + 99 + ... + 2 + 1 = 101 + 101 + ... + 101 + 101 = 100*101

So 1+2+ ... + 99 + 100 = $\frac{100*101}{2}$

Works for any $n$, not just $n=100$
Sum of numbers formula

• Claim: for any \( n \in \mathbb{N} \), \( \sum_{i=0}^{n} i = \frac{n(n+1)}{2} \)

• Proof.
  – Suppose not.
  – Let \( S \) be a set of all numbers \( n' \) such that \( \sum_{i=0}^{n'} i \neq \frac{n'(n'+1)}{2} \). By well-ordering principle, if \( S \neq \emptyset \), then there is the least number \( k \) in \( S \).

  – Case 1: \( k=0 \). But \( \sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2} \). So formula works for \( k=0 \).

  – Case 2: \( k>0 \). Then \( k - 1 \geq 0 \).
    • So \( \sum_{i=0}^{k} i = (\sum_{i=0}^{k-1} i) + k \).
    • As \( k \) is the smallest bad number, the formula works for \( k-1 \).
    • So \( \sum_{i=0}^{k-1} i = \frac{(k-1)k}{2} \).
    • Now, \( \sum_{i=0}^{k} i = (\sum_{i=0}^{k-1} i) + k = \frac{(k-1)k}{2} + k = \frac{k^2-k+2k}{2} = \frac{k^2+k}{2} = \frac{k(k+1)}{2} \).
    • So the formula works for \( k>0 \), too.
  – Contradiction. So \( S \) is empty, thus the formula works for all \( n \in \mathbb{N} \).

Gauss' proof:

\[
1 + 2 + \ldots + 99 + 100 + 100 + 99 + \ldots + 2 + 1 = 101 + 101 + \ldots + 101 + 101 = 100 \times 101
\]

So \( 1+2+\ldots+99+100 = \frac{100 \times 101}{2} \)

Works for any \( n \), not just \( n=100 \).