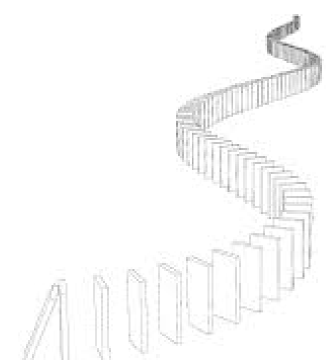
COMP 1002

Logic for Computer Scientists

Lecture 21

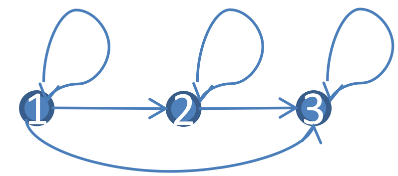


Admin stuff

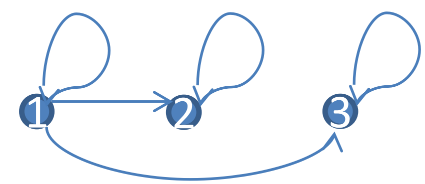
- Assignment 4 is posted
  - Due Tuesday, March 12

Partial and total orders

- A binary relation $R \subseteq A \times A$ is an **order** if R is
  - Reflexive $\forall x \in A \;\; R(x,x)$
  - Anti-symmetric $\forall x,y \in A \;\; R(x,y) \wedge R(y,x) \rightarrow x = y$
  - Transitive $\forall\, x,y,z \in A \;\; R(x,y) \wedge R(y,z) \rightarrow R(x,z)$



- R is a **total order** if $\forall x,y \in A \;\; R(x,y) \vee R(y,x)$
  - That is, every two elements of A are related.
  - E.g. $R_1 = \{(x,y) | x,y \in \mathbb{Z} \wedge x \leq y\}$ is a total order.
  - So is alphabetical order of English words.
  - But not $R_2 = \{(x,y) | x,y \in \mathbb{Z} \wedge x < y\}$
    - not reflexive, so not an order.



- Otherwise, R is a **partial order**.
  - $SUBSETS = \{(A,B) \mid A,B \text{ are sets} \wedge \; A \subseteq B \}$ is a partial order.
    - Reflexive: $\forall A, \;\; A \subseteq A$
    - Anti-symmetric: $\forall\, A,B \;\; A \subseteq B \wedge B \subseteq A \rightarrow A = B$
    - Transitive: $\forall A,B,C \;\; A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$
    - Not total: if A ={1,2} and B ={1,3}, then neither $A \subseteq B$ nor $B \subseteq A$
  - $DIVISORS = \{(\mathrm{x,y}) \mid \; x,y \in \mathbb{N} \wedge x,y \geq 2 \;\; \wedge \exists \mathrm{z} \in \mathbb{N} \;\; y = z \cdot x\}$ is a partial order.

Partial and total orders

- A binary relation $R \subseteq A \times A$ is an **order** if R is
  - Reflexive, Anti-symmetric, Transitive
  - R is a **total order** if $\forall x, y \in A \;\; R(x,y) \vee R(y,x)$
    - $R_1 = \{(x,y) | x,y \in \mathbb{Z} \wedge x \leq y\}$ is a total order.
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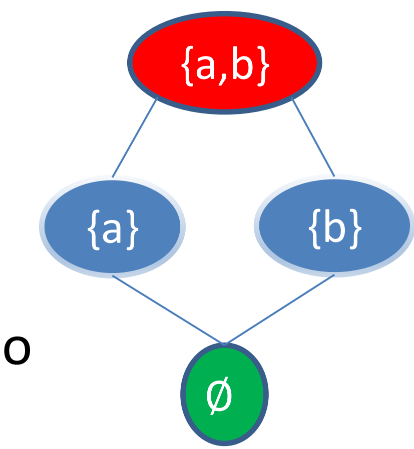
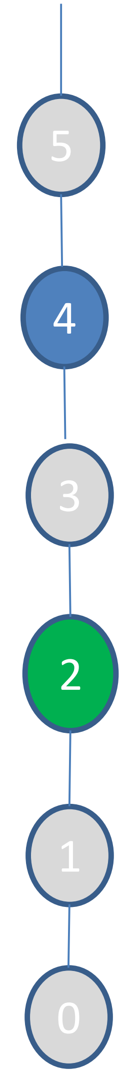
- An order may have **minimal** and **maximal** elements (maybe multiple)
  - $x \in A$ is minimal in R if $\forall y \in A \;\; y \neq x \rightarrow \; \neg R(y,x)$ and maximal if $\forall y \in A \; y \neq x \rightarrow \neg R(x,y)$
  - ∅ is minimal in SUBSETS (its unique minimum); universe is maximal (its unique maximum).
  - All primes are minimal in DIVISORS, and there are no maximal elements.



Well-ordering principle

- **Theorem:** Any non-empty subset of natural numbers contains the least element
  - With respect to the usual total order $x \leq y$
    - There is smallest positive even number. Smallest composite number. Smallest square…
  - In general, if there is a property which is not true for some natural numbers, there is a smallest natural number for which it is not true.
  - Very useful for proofs!
    - We saw it in the proof that there are infinitely many primes



Puzzle: coins

- A not-too-far-away country recently got rid of a penny coin, and now everything needs to be rounded to the nearest multiple of 5 cents…
  - Suppose that instead of just dropping the penny, they would introduce a 3 cent coin.
    - Like British three pence.
  - What is the largest amount that cannot be paid by using only existing coins (5, 10, 25) and a 3c coin?

7c
Any number n >7 can be paid with 3,5,10,25 coins (even just 3 and 5).

Well-ordering principle

- Any non-empty subset of natural numbers contains the least element
  - With respect to the usual total order $x \leq y$
  - Very useful for proofs!
- Coins: $\forall x \in \mathbb{N}$, if x >7 then $\exists\, y, z \in \mathbb{N}$ such that x = 3y+5z. So any amount >7 can be paid with 3s and 5s.
  - Suppose, for the sake of contradiction, that there are amounts greater than 7 which cannot be paid with 3s and 5s.
  - *Take a set S of all such amounts. Since $S \subseteq \mathbb{N}$, and we assumed that $S \neq \emptyset$, by well-ordering principle S has the least element. Call it n.*
  - Now, look at n-3; it cannot be paid by 3s and 5s either.
  - Since n is the least element of S, $n - 3 \leq 7 < n$
  - 3 cases:
    - n-3 = 7. Then n=10=2*5.
    - n-3 = 6. Then n=9=3*3
    - n-3 = 5. Then n=8=3+5.
  - In all three cases, got a contradiction.
  - Therefore, for every $x \in \mathbb{N}$, if x >7 then x=3y+5z for some $y, z \in \mathbb{N}$.

Sums, products and sequences

- How to write long sums, e.g., 1+2+… (n-1)+n concisely?
  - Sum notation ("sum from 1 to n"): $\sum_{i=1}^{n} i = 1 + 2 + \ \ldots + n$
    - If n=3, $\sum_{i=1}^{3} i$ = 1+2+3=6.
    - The name "i" does not matter. Could use another letter not yet in use.
- In general, let $f: \mathbb{Z} \to \mathbb{R},\ m, n \in \mathbb{Z}, m \leq n$.
  - $\sum_{i=m}^{n} f(i) = \ f(m) + f(m+1) + \cdots + f(n)$
    - If m=n, $\sum_{i=m}^{n} f(i)$ =f(m)=f(n).
    - If n=m+1, $\sum_{i=m}^{n} f(i)$ = f(m)+f(m+1)
    - If n>m, $\sum_{i=m}^{n} f(i)$ = $(\sum_{i=m}^{n-1} f(i)) + f(n)$
    - Example: $f(x) = x^2$. $2^2 + 3^2 + 4^2 = \sum_{i=2}^{4} i^2 = 29$
- Similarly for product notation (product from m to n):
  - $\Pi_{i=m}^{n} f(i) = f(m) \cdot f(m+1) \cdot \ldots \cdot f(n)$ = $(\Pi_{i=m}^{n-1} f(i)) \cdot f(n)$
  - For $f(x) = x,\ 2 \cdot 3 \cdot 4 \ = \Pi_{i=2}^{4} i = 24$
  - $1 \cdot 2 \cdot \ldots \cdot n = \Pi_{i=1}^{n} i = n!$ (n factorial)

Sum of numbers formula

Gauss' proof:
1 + 2 + … + 99 + 100 +
100 + 99 + … + 2 + 1 =
101 + 101+ … +101 + 101 =100*101
So 1+2+ … + 99 + 100 = $\frac{100*101}{2}$
Works for any n, not just n=100

- Claim: for any n$\in \mathbb{N}$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$
- Proof.
  - Suppose not.
  - Let S be a set of all numbers n' such that $\sum_{i=0}^{n'} i \neq \frac{n'(n'+1)}{2}$. By well-ordering principle, if $S \neq \emptyset$, then there is the least number k in S.
  - Case 1: k=0. But $\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$. So formula works for k=0.
  - Case 2: k>0. Then $k-1 \geq 0$.
    - So $\sum_{i=0}^{k} i = (\sum_{i=0}^{k-1} i)$ +k.
    - As k is the smallest bad number, the formula works for k-1.
    - So $\sum_{i=0}^{k-1} i = \frac{(\mathrm{k}-1)\mathrm{k}}{2}$
    - Now, $\sum_{i=0}^{k} i = (\sum_{i=0}^{k-1} i)$ +k = $\frac{(\mathrm{k}-1)\mathrm{k}}{2} + \mathrm{k} = \frac{\mathrm{k}^2-\mathrm{k}+2\mathrm{k}}{2} = \frac{\mathrm{k}^2+k}{2} = \frac{k(k+1)}{2}$
    - So the formula works for k>0, too.
  - Contradiction. So S is empty, thus the formula works for all $n \in \mathbb{N}$.

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