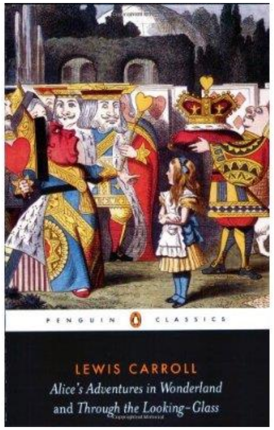
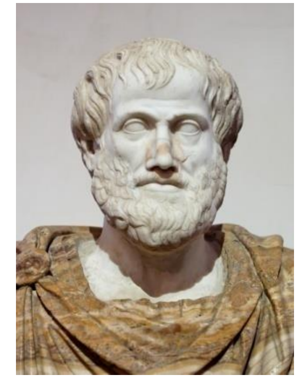




COMP 1002

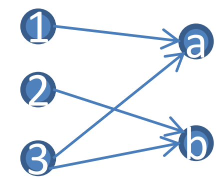
Logic for Computer Scientists

Lecture 20

Relations

- **A relation** is a subset of a Cartesian product of sets.
  - If of two sets (set of pairs), call it a **binary** relation.
  - Of 3 sets (set of triples), **ternary**. Of k sets (set of tuples), **k-ary**

  - A={1,2,3}, B={a,b}
    - $A \times B = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$
    - R = {(1,a), (2,b),(3,a), (3,b)} is a relation. So is R={(1,b)}.
  - A={1,2},
    - $A \times A = \{(1,1),(1,2),(2,1),(2,2)\}$
    - R={(1,1), (2,2)} (all pairs (x,y) where x=y)
    - R={(1,1),(1,2),(2,2)} (all pairs (x,y) where $x \leq y$).
  - A=PEOPLE
    - COUPLES ={(x,y) | Loves(x,y)}
    - PARENTS ={(x,y) | Parent(x,y)}
  - A=PEOPLE, B=DOGS, C=PLACES
    - WALKS = {(x,y,z) | x walks y in z}
      - Jane walks Buddy in Bannerman park.

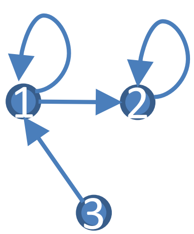


Bipartite graph of $R \subseteq \{1,2,3\} \times \{a,b\}$ Arrows for each pair $(x,y) \in R$

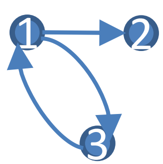
Graphs of binary relations

- **A (directed) graph (digraph)** of a binary relation $R \subseteq A \times A$ is a diagram consisting of
  - |A| points, with a point (often drawn as a circle with a label, called a vertex or a node) for each element of A
  - An arrow (called an edge, an arc or a link) from point x to point y for each $(x, y) \in R$
    - We draw a loop with an arrow for each $x \in A$ such that $(x, x) \in R$

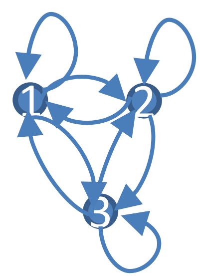
- Let A={1,2,3}



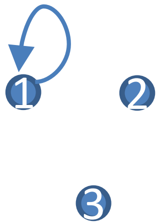
$R = \{(1,1), (1,2), (2,2), (3,1)\}$



$R = \{(1,2), (1,3), (3,1)\}$



$R = A \times A$

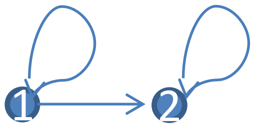


$R = \{(1,1)\}$

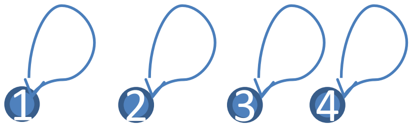
- This is a different graph than plotting a function on plane!

Reflexive relations

- A binary relation $R \subseteq A \times A$ is
  - **Reflexive** if $\forall x \in A, R(x,x)$
    - Every x is related to itself.
      - So on the graph, every vertex has a loop
    - E.g. A={1,2}, R_1 = { (1,1), (2,2), (1,2)}
    - On A = $\mathbb{Z}$, $R_2 = \{(x,y)|x = y\}$ is reflexive
    - But not $R_3 = \{(x,y)|\, x < y\}$



R_1



R_2

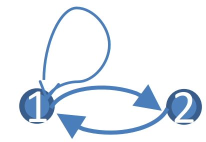
Anti-reflexive relations

R_6 ={(1,2)}

- A binary relation $R \subseteq A \times A$ is
  - **Anti-reflexive** if $\forall x \in A, \neg R(x,x)$
    - Graph of R has no loops.
    - E.g. A={1,2}, $R_6 = \{(1,2)\}$
      - but not $R_1 = \{ (1,1), (2,2), (1,2)\}$ (reflexive)
      - nor $R_7 = \{(1,1), (1,2)\}$ (neither)
    - For $A = \mathbb{Z}$, not $R_2 = \{(x,y) | x = y\}$
      - Nor $R_4 = \{(x,y) | x \equiv y \ mod \ 3 \}$
    - But $R_3 = \{(x,y) | \ x < y\}$ is anti-reflexive.
      - So are $R_5 = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \ | \ x + 1 = y\}$
      - And PARENT = $\{(x,y) \in PEOPLE \times PEOPLE \ | x \ \ is \ a \ parent \ of \ y\}$
  - A relation R can be neither reflexive nor anti-reflexive.

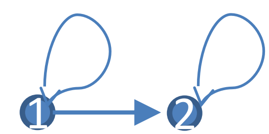
Symmetric and anti-symmetric relations

- A binary relation $R \subseteq A \times A$ is
  - **Symmetric** iff $\forall x, y \in A,\ (x, y) \in R \leftrightarrow (y, x) \in R$
    - For every arrow (except loops) another goes the opposite way
    - R_1 and R_3 above are not symmetric. R_2 is.
    - $R_8 = \{(1,2), (2,1), (1,1)\}$ is symmetric
    - A = $\mathbb{Z}$, $R_4 = \{(x, y) | x \equiv y \ mod\ 3 \}$ is symmetric.



R_8

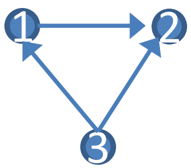
  - **Anti-symmetric** iff $\forall x, y \in A, (x, y) \in R \land (y, x) \in R \rightarrow x = y$
    - For every arrow, there is no arrow the other way. Loops OK.
    - $R_1, R_3, R_5, R_6, R_7, PARENT$ are anti-symmetric.
    - R_4 is not.
    - R_2 is both symmetric and anti-symmetric.
    - $R_8 = \{(1,2), (2,1), (1,3)\}$ is neither symmetric nor anti-symmetric.



R_1

Transitive relations

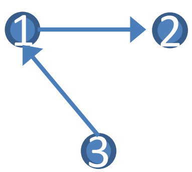
- A binary relation $R \subseteq A \times A$ is **transitive** if
  $\forall x, y, z \in A,\ (x,y) \in R \wedge (y,z) \in R \rightarrow (x,z) \in R$
  – In the graph of R, if there is a way to get from x to z by following a sequence of edges (arrows), then there is a way to get from x to z in one step.
  – R_1, R_2, R_3, R_4 are all transitive.
  – $R_5 = \{(x,y) | x,y \in \mathbb{Z} \wedge x+1=y\}$ is not transitive.
  – PARENT = $\{(x,y) | x,y \in PEOPLE \wedge x\ \ is\ a\ parent\ of\ \ y\}$ is not.



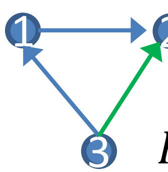
- A **transitive closure** of a relation R is the relation

$$R^* = \{(x,z) |\ \exists k \in \mathbb{N}\ \ \exists y_0, \ldots, y_k \in A$$
$$(x = y_0 \wedge z = y_k \wedge \forall i \in \{0, \ldots, k-1\}\ R(y_i, y_{i+1})\}$$

  – That is, the graph of R^* has all vertices and edges of R, plus a direct arrow x to z whenever there is a way in the graph of R to get from x to z following a sequence of arrows.
  – If $R = \{(3,1), (1,2)\}$, then there is a way in R to get from 3 to 2 via 1, so in R* there is a direct arrow from 3 to 2.
    - So the transitive closure of {(3,1),(1,2)} is $R^* = \{(3,1), (1,2), (3,2)\}$



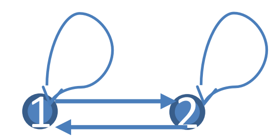
R

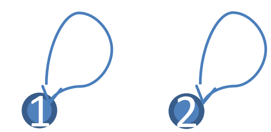


R^*

Equivalence

- A binary relation $R \subseteq A \times A$ is an **equivalence** if R is reflexive, symmetric and transitive.
  - E.g. A={1,2}, $R = \{(1,1),(2,2)\}$ or $R = A \times A$
  - Not $R_1 = \{$ (1,1), (2,2), (1,2)$\}$ nor $R_3 = \{(x,y) |\ x < y\}$
  - On A = $\mathbb{Z}$, $R_2 = \{(x,y) | x = y\}$ is an equivalence
  - So is $R_4 = \{(x,y) | x \equiv y \ mod\ 3 \}$
    - Reflexive: $\forall x \in \mathbb{Z},\ x \equiv x \ mod\ 3$
    - Symmetric: $\forall x, y \in \mathbb{Z},\ x \equiv y \text{ mod } 3 \rightarrow y \equiv x \ mod\ 3$
    - Transitive: $\forall x, y, z \in \mathbb{Z},\ x \equiv y \text{ mod } 3 \wedge y \equiv z \ mod\ 3 \rightarrow x \equiv z \ mod\ 3$

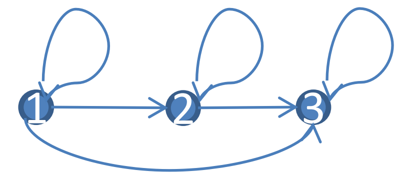




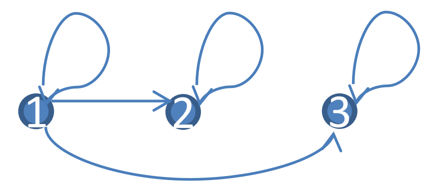
- An equivalence relation partitions A into **equivalence classes**:
  - Intersection of any two equivalence classes is $\emptyset$
  - Union of all equivalence classes is A.
  - R_4: $\mathbb{Z} = \{x\ | x \equiv 0 \ mod\ 3\} \cup \{\text{x} \mid \text{x} \equiv 1 \ mod\ 3\} \cup \{x\ | x \equiv 2 \ mod\ 3\}$
  - $R = A \times A$ gives rise to a single equivalence class. $R = \{(1,1),(2,2)\}$ on A ={1,2} to two equivalence classes.

Partial and total orders

- A binary relation $R \subseteq A \times A$ is an **order** if R is reflexive, anti-symmetric and transitive.
  - R is a **total order** if $\forall x, y \in A \;\; R(x,y) \vee R(y,x)$
    - That is, every two elements of A are related.
    - E.g. $R_1 = \{(x,y) | x, y \in \mathbb{Z} \wedge x \leq y\}$ is a total order.
    - So is alphabetical order of English words.
    - But not $R_2 = \{(x,y) | x, y \in \mathbb{Z} \wedge x < y\}$
      - not reflexive, so not an order.



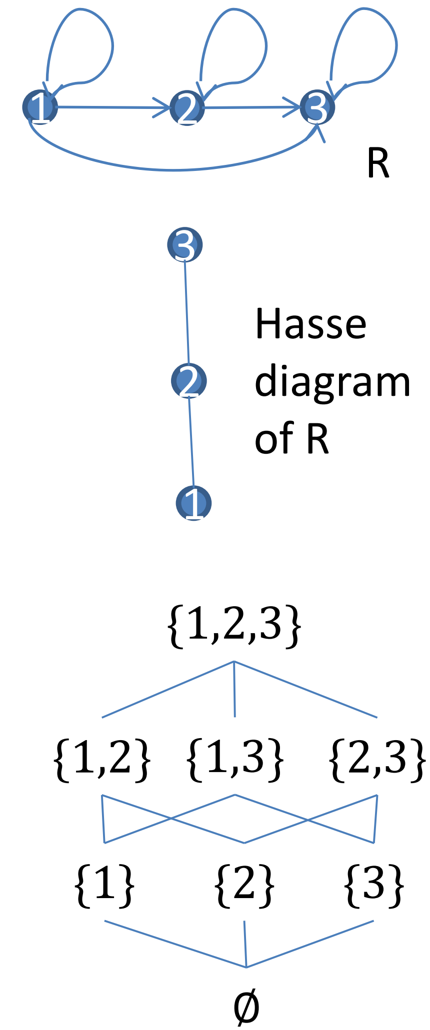
  - Otherwise, R is a **partial order**.
    - $SUBSETS = \{(A,B) \mid A, B \text{ } are \text{ } sets \wedge \; A \subseteq B \}$ is a partial order.
      - Reflexive: $\forall A, \;\; A \subseteq A$
      - Anti-symmetric: $\forall \, A, B \;\; A \subseteq B \wedge B \subseteq A \rightarrow A = B$
      - Transitive: $\forall A, B, C \;\; A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$
      - Not total: if A ={1,2} and B ={1,3}, then neither $A \subseteq B$ nor $B \subseteq A$
    - $DIVISORS = \{(x,y) | \; x, y \in \mathbb{N} \wedge x, y \geq 2 \;\; \wedge \exists z \in \mathbb{N} \;\; y = z \cdot x\}$ is a partial order.
    - PARENT is not an order. But ANCESTOR would be, if defined so that each person is an ancestor of themselves. It is a partial order.



- An order may have **minimal** and **maximal** elements (maybe multiple)
  - $x \in A$ is minimal in R if $\forall y \in A \;\; y \neq x \rightarrow \; \neg R(y,x)$
    - and maximal if $\forall y \in A \; y \neq x \rightarrow \neg R(x,y)$
  - Ø is minimal in SUBSETS (its unique minimum); universe is maximal (its unique maximum).
  - All primes are minimal in DIVISORS, and there are no maximal elements.

Hasse diagram

- A Hasse diagram is a way to draw a (partial or total) order (more precisely, its “transitive reduction”: opposite of transitive closure) without drawing loops or edges that have to be there by transitivity or reflexivity.
  - draw minimal elements on the bottom, and then go up
  - don’t draw arrowheads (assumed arrow direction is always upwards).
  - R={ $(x, y) \in \{1,2,3\} \times \{1,2,3\} \mid x \leq y\}$
    - On the Hasse diagram of R, only draw edges (1,2) and (2,3), as all the rest follow by reflexivity and transitivity. 1 is the minimal (bottom), 3 maximal (top).
  - $SUBSETS = \{(A, B) \mid A, B \text{ are sets} \wedge A \subseteq B\}$
    - Let universe be {1,2,3}
    - Hasse diagram of SUBSETS over {1,2,3}:



Tower of Hanoi game

- Rules of the game:
  - Start with all disks on the first peg.
  - At any step, can move a disk to another peg, as long as it is not placed on top of a smaller disk.
  - Goal: move the whole tower onto the second peg.
- Question: how many steps are needed to move the tower of 8 disks? How about n disks?