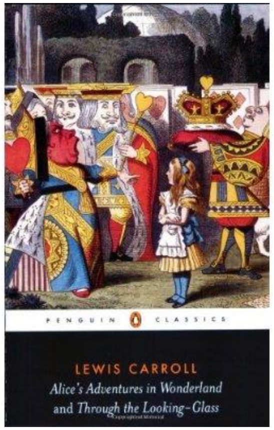
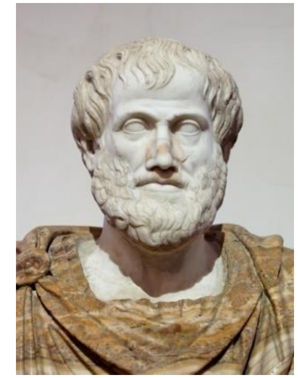




COMP 1002

Logic for Computer Scientists

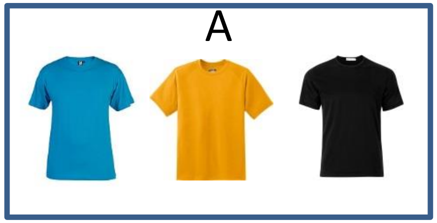
Lecture 18

B 5 2 J

Cartesian products

- **Cartesian product** of A and B is a set of all pairs of elements with the first from A, and the second from B:
  - A x B = $\{(x, y) | x \in A,\ y \in B\}$
  - A={1,2,3}, B={a,b}
  - $A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$
  - A={1,2}, $A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$

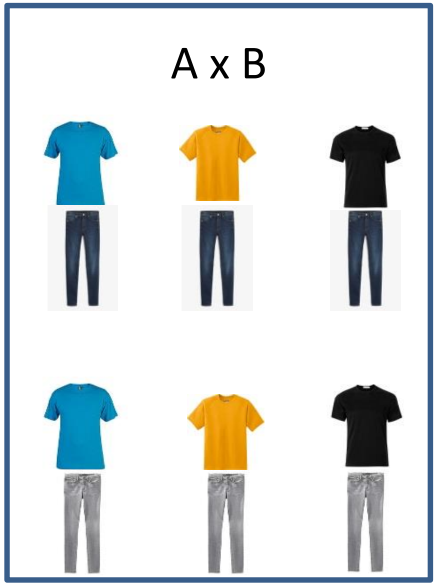
	a	b
1	(1,a)	(1,b)
2	(2,a)	(2,b)
3	(3,a)	(3,b)



- Order of pairs does not matter, order within pairs does: $A \times B \neq B \times A$.

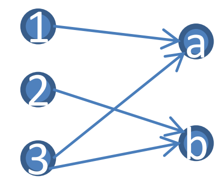
- Number of elements in $A \times B$ is $|A \times B| = |A| \cdot |B|$

- Can define the Cartesian product for any number of sets:
  - $A_1 \times A_2 \times \cdots \times A_k = \{(x_1, x_2, \ldots x_k) |\ x_1 \in A_1 \ldots x_k \in A_k\}$
  - $A = \{1,2,3\}$, B ={a,b}, C={3,4}
  - $A \times B \times C = \{(1, a, 3), (1, a, 4),\ (1, b, 3), (1, b, 4),$
    $(2, a, 3), (2, a, 4),\ (2, b, 3), (2, b, 4),$
    $(3, a, 3), (3, a, 4),\ (3, b, 3), (3, b, 4)\}$

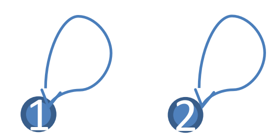


Relations

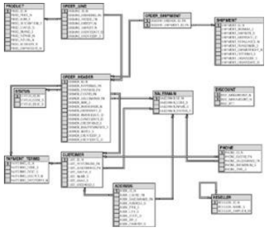
- **A relation** is a subset of a Cartesian product of sets.
  - If of two sets (set of pairs), call it a **binary** relation.
  - Of 3 sets (set of triples), **ternary**.   Of k sets (set of k-tuples), **k-ary**
  - A={1,2,3}, B={a,b}
    - $A \times B = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$
    - R = {(1,a), (2,b),(3,a), (3,b)} is a relation. So is R={(1,b)}.
  - A={1,2},
    - $A \times A \ = \{(1,1),(1,2),(2,1),(2,2)\}$
    - R={(1,1), (2,2)} (all pairs (x,y) where x=y)
    - R={(1,1),(1,2),(2,2)} (all pairs (x,y) where $x \leq y$).
  - A=PEOPLE
    - COUPLES ={(x,y) | Loves(x,y)}
    - PARENTS ={(x,y) | Parent(x,y)}
  - A=PEOPLE, B=DOGS, C=PLACES
    - WALKS = {(x,y,z) | x walks y in z}
      - Jane walks Buddy in Bannerman park.



Graph of R (bipartite)



Graph of {(1,1),(2,2)}



Databases and predicates

Relation R	**Predicate P**
A set of tuples	True/false on a given tuple
R= $\{(x_1, \ldots, x_k) \mid P(x_1, \ldots, x_k)$ is true $\}$	$P(x_1, \ldots, x_k) \equiv (x_1, \ldots, x_k) \in R$

- In a database, store relations as tables.
- Then ask queries as predicate logic formulas
  - Return the set of all database elements satisfying the formula.

ProfData

	A	B	C
1	Manrique	Mata-Montero	EN-2033
2	Sharene	Bungay	ER-6032
3	Antonina	Kolokolova	ER-6033

CourseData

	A	B	C	D	E
1	COMP1000	MWF	11:00-11:50	EN-1054	Mata-Montero
2	COMP1001	MWF	12:00-12:50	EN-2040	Bungay
3	COMP1002	MTR	13:00-13:50	EN-2007	Kolokolova

  - "Return first names of all profs who teach MWF "
  - $Q(fn)$:

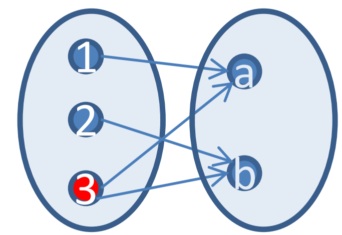
    $\exists\, ln\, \exists\, o\;\; ProfData(fn, ln, o)$

    $\wedge\, \exists c, t, r\;\, CourseData(c, \text{"MWF"}, \text{t,r,ln})$

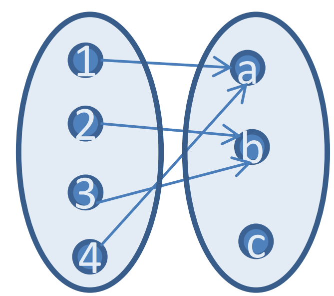
Functions

- **A function** $f: X \to Y$ is a relation on $X \times Y$ such that for every $x \in X$ there is at most one $y \in Y$ for which (x, y) is in the relation.

- Usual notation: $f(x) = y$, where
  - y is an **image** of x under f.
  - X is the **domain** of f
  - Y is the **codomain** of f
  - **Range** of f (**image** of X under f):
    - $\{y \in Y \mid \exists x \in X, f(x) = y\}$
  - **Preimage** of a given $y \in Y$:
    - $\{x \in X \mid f(x) = y\}$
      - Preimage of b is {2,3}.



This R is not a function

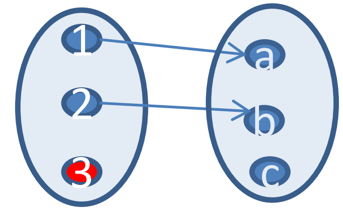


This R is a function with domain {1,2,3,4}, codomain {a,b,c} and range {a,b}

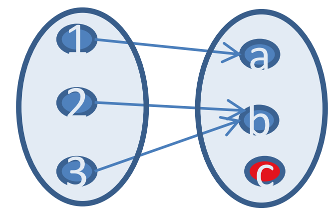
Functions

- **A function** $f: X \to Y$ is
  - **Total**: $\forall x \in X\ \exists y \in Y\ f(x) = y$
    - f: $\mathbb{Z} \to \mathbb{Z}$
    - $f(x) = x + 1$ is total.
    - $f(x) = \frac{100}{x}$ is not total.
    - A function that is not necessarily total is **partial**
  - **Onto**: $\forall y \in Y\ \exists x \in X\ f(x) = y$
    - $f(x) = x + 1$ is onto over $\mathbb{Z}$, but not over $\mathbb{N}$
    - $f(x) = 5x$ is not onto (over $\mathbb{Z}$)
  - **One-to-one:** $\forall x_1, x_2 \in X\ f(x_1) = f(x_2) \to x_1 = x_2$
    - $f(x) = x + 1$ is one-to-one.
    - $f(x) = x^2$ is not one-to-one
  - **Bijection**: both one-to-one and onto.
    - $f(x) = x + 1$ is a bijection over $\mathbb{Z}$.

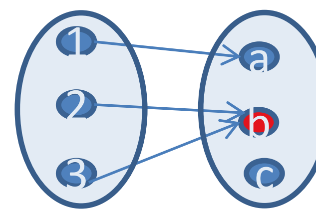
Not total



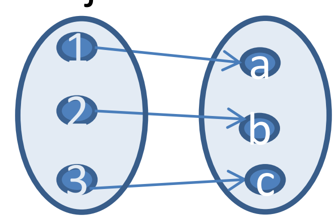
Not onto



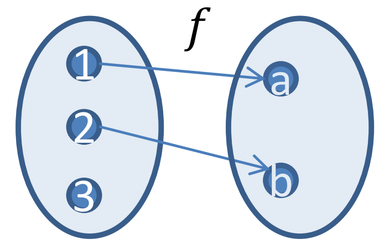
Not one-to-one

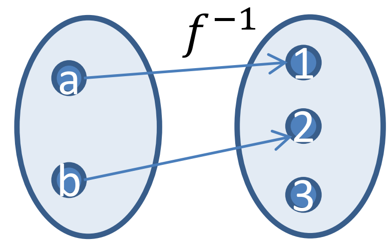


Bijection



Functions



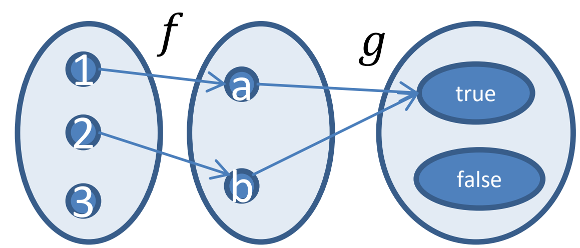


- An **inverse** of f is a function $f^{-1}: Y \to X$, such that $f^{-1}(y) = x$ iff $f(x) = y$
  - $f(x) = x + 1, f^{-1}(y) = y - 1$
  - Only one-to-one functions have an inverse

Functions

- **Composition** of functions $f: X \to Y$ and $g: Y \to Z$ is a function $g \circ f: X \to Z$ such that

$$(g \circ f)(x) = g(f(x))$$



  - $f(x) = \frac{x}{5}, \; g(x) = \lceil x \rceil$, over $\mathbb{R}$
    - $\lceil x \rceil$ is ceiling: x rounded up to nearest integer.
  - $(g \circ f)(x) = g(f(x)) = \left\lceil \frac{x}{5} \right\rceil$
  - $(f \circ g)(x) = f(g(x)) = \frac{\lceil x \rceil}{5}$
  - $(g \circ f)(12.5) = \lceil 2.5 \rceil = 3$. $(f \circ g)(12.5) = 13/5 = 2.6$
    - Order matters!

Puzzle: the barber club

- In a certain barber's club,
  - Every member has shaved at least one other member
  - No member shaved himself
  - No member has been shaved by more than one member
  - There is a member who has never been shaved.

- *Question: how many barbers are in this club?*