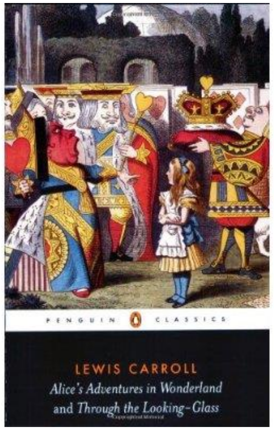
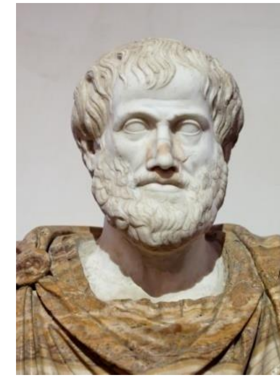




COMP 1002

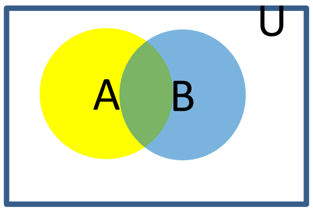
Logic for Computer Scientists

Lecture 17

B 5 2 J

Enrollment puzzle

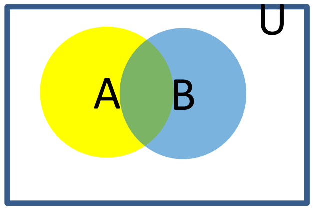
- There are 160 students in 1000 this semester
- There are 105 students in 1002
- The total number of students in either of these two courses is 200
- How many students are in both 1000 and 1002?



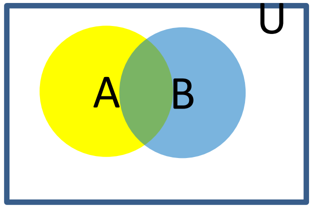
Enrollment puzzle

- There are 160 students in 1000 this semester. There are 105 students in 1002. The total number of students in either of these two courses is 200. How many students are in both 1000 and 1002?

- Let A be a set of students in 1000, and B the set of students in 1002. |A| = 160 and |B| = 105.
- The number of students in either 1000 or 1002 is $|A \cup B| = 200$. We want to know $|A \cap B|$.
- If we take |A|+|B|, we would count the students in both 1000 and 1002 twice.
- So the number of students in both is the number of double-counted students:
- $|A \cap B| = |A| + |B| - |A \cup B|$= 160+105-200=65.



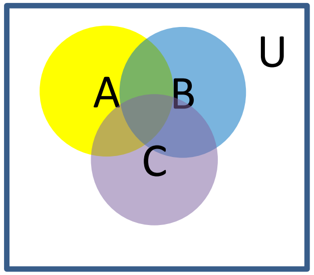
Rule of inclusion-exclusion



- Let A and B be two sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

  - Proof idea: notice that elements in $|A \cap B|$ are counted twice in |A|+|B|, so need to subtract one copy.
  - If A and B are disjoint, then $|A \cup B| = |A| + |B|$
  - If there are 160 students in COMP 1000, 105 in COMP 1002, and 65 of them are in both, then the total number of students in 1000 or 1002 is 160+105-65=200.



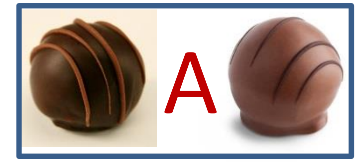
- For three sets (and generalizes)
- $|A \cup B \cup C| = |A| + |B| + |C|$

$$-|A \cap B| - |A \cap C| - |B \cap C|$$

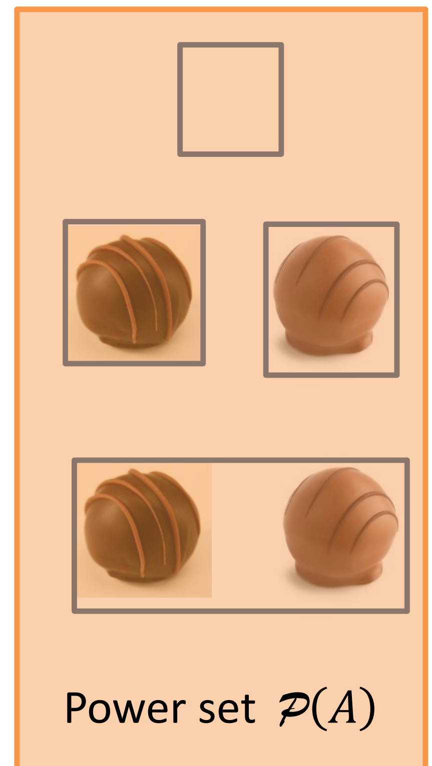
$$+|A \cap B \cap C|$$

Power sets

- A **power set** of a set A, $\mathcal{P}(A)$, is a set of all subsets of A.
  - Think of sets as boxes of elements.
  - A subset of a set A is a box with elements of A (maybe all, maybe none, maybe some).
  - Then $\mathcal{P}(A)$ is a box containing boxes with elements of A.
  - When you open the box $\mathcal{P}(A)$, you don't see chocolates (elements of A), you see boxes.
  - A={1,2}, $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$
  - $A = \emptyset, \quad \mathcal{P}(A) = \{\emptyset\}$.
    - They are not the same! There is nothing in A, and there is one element, an empty box, in $\mathcal{P}(A)$
- If A has n elements, then $\mathcal{P}(A)$ has 2^n elements.



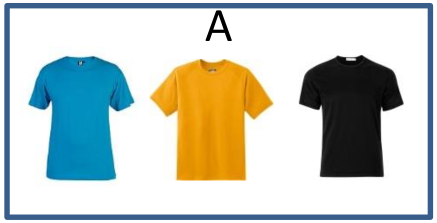
Subsets of A:



Cartesian products

- **Cartesian product** of A and B is a set of all pairs of elements with the first from A, and the second from B:
  - A x B = $\{(x, y) | x \in A,\ y \in B\}$
  - A={1,2,3}, B={a,b}
  - $A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$
  - A={1,2}, $A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$

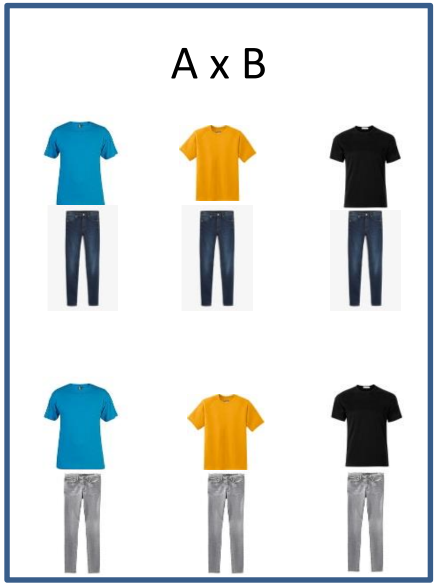
	a	b
1	(1,a)	(1,b)
2	(2,a)	(2,b)
3	(3,a)	(3,b)



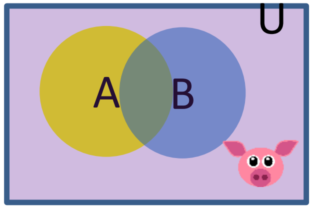
- Order of pairs does not matter, order within pairs does: $A \times B \neq B \times A$.

- Number of elements in $A \times B$ is $|A \times B| = |A| \cdot |B|$

- Can define the Cartesian product for any number of sets:
  - $A_1 \times A_2 \times \cdots \times A_k = \{(x_1, x_2, \ldots x_k) |\ x_1 \in A_1 \ldots x_k \in A_k\}$
  - $A = \{1,2,3\}$, B ={a,b}, C={3,4}
  - $A \times B \times C = \{(1, a, 3), (1, a, 4), (1, b, 3), (1, b, 4), (2, a, 3), (2, a, 4), (2, b, 3), (2, b, 4), (3, a, 3), (3, a, 4), (3, b, 3), (3, b, 4)\}$



Proofs with sets



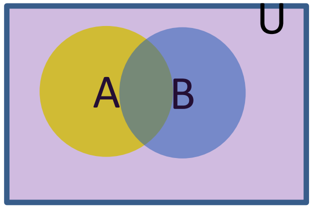
- Two ways to describe the purple area
- $\overline{A \cup B}$, $\overline{A} \cap \overline{B}$
  - $x \in \overline{A \cup B}$ when $x \notin A \cup B$
  - This happens when $x \notin A \wedge x \notin B$.
  - So $x \in \overline{A} \cap \overline{B}$. Since we picked an arbitrary x, then $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$

  - Not quite done yet… Now let $x \in \overline{A} \cap \overline{B}$
  - Then $x \in \overline{A} \wedge x \in \overline{B}$. So $x \notin A \wedge x \notin B$.
  - $x \notin A \wedge x \notin B \equiv \neg(x \in A \vee x \in B)$. So $x \notin A \cup B$. Thus $x \in \overline{A \cup B}$.
  - Since x was an arbitrary element of $\overline{A} \cap \overline{B}$, then $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

  - Therefore $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Laws of set theory

- Two ways to describe the purple area
  - $\overline{A \cup B} = \overline{A} \cap \overline{B}$



- By similar reasoning,
  - $\overline{A \cap B} = \overline{A} \cup \overline{B}$

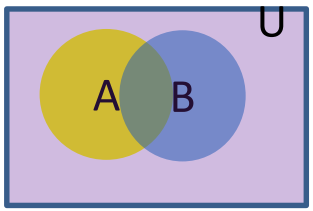
- Does this remind you of something?...
  - $\neg(p \vee \mathrm{q}) \equiv \neg p \ \wedge \neg\, q$
  - DeMorgan's law works in set theory!
  - What about other equivalences from logic?

More useful equivalences

- For any formulas A, B, C:
  - $A \lor \neg A \equiv True$ $\quad$ $A \land \neg A \equiv False$
  - $True \lor A \equiv True$. $\quad$ $True \land A \equiv A$
  - $False \lor A \equiv A$. $\quad$ $False \land A \equiv False$
  - $A \lor A \equiv A \land A \equiv A$
- Also, like in arithmetic (with ∨ as +, ∧ as *)
  - $A \lor B \equiv B \lor A \quad and \quad (A \lor B) \lor C \equiv A \lor (B \lor C)$
  - Same holds for ∧.
  - Also, $(A \lor B) \land C \equiv (A \land C) \lor (B \land C)$
- And unlike arithmetic
  - $(A \land B) \lor C \equiv (A \lor C) \land (B \lor C)$


Propositions vs. sets

Propositional logic	Set theory
Negation $\neg p$	Complement $\overline{A}$
$AND \ \ p \wedge q$	Intersection $A \cap B$
OR $p \vee q$	Union $A \cup B$
FALSE	Empty set ∅
TRUE	Universe U



More useful equivalences

- For any formulas A, B, C:
  - $A \lor \neg A \equiv True$ $\quad$ $A \land \neg A \equiv False$
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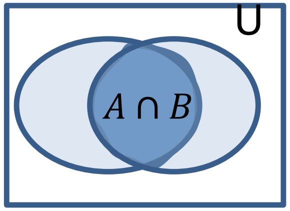
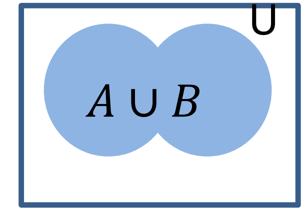
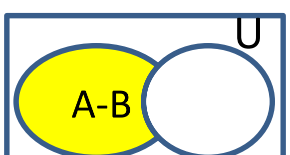
  - $A \lor A \equiv A \land A \equiv A$
- Also, like in arithmetic (with ∨ as +, ∧ as *)
  - $A \lor B \equiv B \lor A \quad and \quad (A \lor B) \lor C \equiv A \lor (B \lor C)$
  - Same holds for ∧.
  - Also, $(A \lor B) \land C \equiv (A \land C) \lor (B \land C)$
- And unlike arithmetic
  - $(A \land B) \lor C \equiv (A \lor C) \land (B \lor C)$

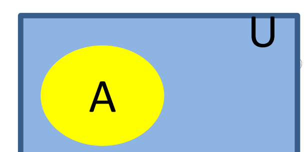


Laws of set theory



- For any sets A, B, C:
  - $A \cup \overline{A} = U$ $\quad\quad$ $A \cap \overline{A} = \emptyset$
  - $U \cup A = U.$ $\quad\quad$ $U \cap A = A$
  - $\emptyset \cup A = A.$ $\quad\quad$ $\emptyset \cap A = \emptyset$
  - $A \cup A = A \cap A = A$
- Also, like in arithmetic (with ∨ as +, ∧ as *)
  - $A \cup B = B \cup A \quad and \quad (A \cup B) \cup C = A \cup (B \cup C)$
  - Same holds for ∩.
  - Also, $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
- And unlike arithmetic
  - $(A \cap B) \cup C \equiv (A \cup C) \cap (B \cup C)$





Boolean algebra

- The "algebra" of both propositional logic and set theory is called **Boolean algebra** (as opposed to algebra on numbers).

Propositional logic	Set theory	Boolean algebra
Negation $\neg p$	Complement $\overline{A}$	$\overline{a}$
AND $p \wedge q$	Intersection $A \cap B$	$a \cdot b$
OR $p \vee q$	Union $A \cup B$	$a + b$
FALSE	Empty set $\emptyset$	0
TRUE	Universe U	1

Axioms of Boolean algebra

1. $a + b = b + a,$ $\qquad a \cdot b = b \cdot a$
2. (a+b)+c=a+(b+c) $\qquad (\mathrm{a} \cdot b\,) \cdot c = a \cdot (b \cdot c)$
3. $a + (b \cdot c) = (a + b) \cdot (a + c)$
   $a \cdot (b + c) = (a \cdot b) + (a \cdot c\,)$
4. There exist distinct elements 0 and 1 (among underlying set of elements B of the algebra) such that for all $a \in B$,
   $$a + 0 = a \qquad a \cdot 1 = a$$
5. For each $a \in B$ there exists an element $\overline{a} \in B$ such that
   $$a + \overline{a} = 1 \qquad a \cdot \overline{a} = 0$$

How about DeMorgan, etc? They can be derived from the axioms!