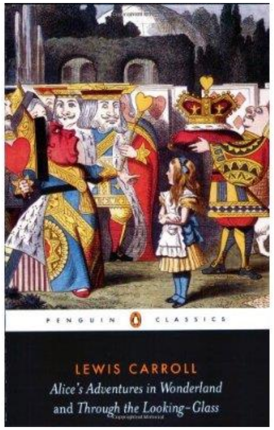
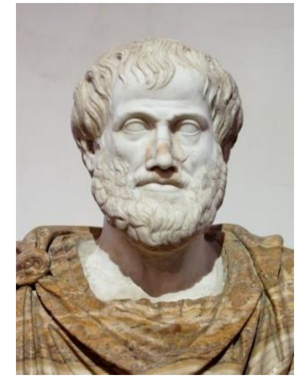




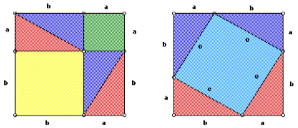
COMP 1002

Logic for Computer Scientists

Lecture 16

B 5 2 J


HAPPY NEW YEAR

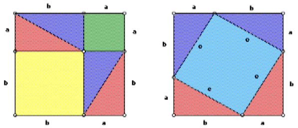


Proof by contradiction

– To prove $\forall x \; F(x)$, prove $\forall x \, \neg F(x) \rightarrow FALSE$

- Universal instantiation: “let n be an arbitrary element of the domain S of $\forall x$”
- Suppose that $\neg F(n)$ is true.
- Derive a contradiction.
- Conclude that $F(n)$ is true.
- By universal generalization, $\forall x \, F(x)$ is true.

☐ (Done).



Infinitely many primes

- *Definition:* A natural number is prime iff it is divisible only by 1 and itself. That is, n is prime iff
  $\forall z \in \mathbb{N}\ ((\exists w \in \mathbb{N}\ n = zw) \rightarrow (z = 1 \lor z = n\ ))$

- *Theorem*: There are infinitely many primes.
  - $\forall x \in \mathbb{N}\ Prime(x) \rightarrow \exists y \in \mathbb{N} y > x \land Prime(y)$

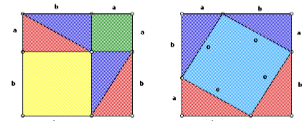
- $Prime(x)$ is a shorthand for $\forall z \in \mathbb{N}(\exists w \in \mathbb{N}\ x = zw) \rightarrow z = 1 \lor z = x$
- To say "infinitely many" we can write that no matter what element of the domain we take, there is always a larger one that has the property we are interested in (in this case, a prime).

Infinitely many primes

- *Theorem*: There are infinitely many primes.
  - $\forall x \in \mathbb{N}\ Prime(x) \rightarrow \exists y \in \mathbb{N}\ y > x \wedge Prime(y)$

- *Proof*:
  - Assume, for the sake of contradiction, that the statement of the theorem is false: that is,

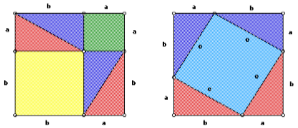
  $\exists x \in \mathbb{N} Prime(x) \wedge (\forall y \in \mathbb{N} y \leq x \vee \neg Prime(y))$
  - Call this number n (universal instantiation of $\forall x$)
  - Now consider the number $\mathrm{N} = (2 \cdot 3 \cdot \ldots \cdot n) + 1$
  - Now we have 2 cases.
    - Either N is a prime, in which case we are done since we found a prime larger than n, contradicting our assumption.
    - or N is not prime.



Infinitely many primes

- *Theorem*: There are infinitely many primes.
  - $\forall x \in \mathbb{N}\ Prime(x) \rightarrow \exists y \in \mathbb{N}\ y > x \land Prime(y)$

- *Proof (continued)*:
  - Consider the number $\mathrm{N} = (2 \cdot 3 \cdot \ldots \cdot n) + 1$
  - Case 2: suppose N is not prime, that is, for some $k, \mathrm{q} \in \mathbb{N},\ N = kq$, where $k \neq 1$ and $k \neq N$. Take smallest such k.
    - Since $N \equiv 1\ mod\ d$ for all $d \leq n,$ this k is not divisible by any $d \leq n$, and $k > n$
    - So since k is the smallest factor of N, k itself must be prime.
    - Therefore, there exists a prime number $k > n$ by existential instantiation.



Infinitely many primes

- *Theorem*: There are infinitely many primes.
  - $\forall x \in \mathbb{N}\ Prime(x) \rightarrow \exists y \in \mathbb{N}\ y > x \wedge Prime(y)$

- *Proof (continued)*:
  - We showed that both cases of N being prime and not being prime give us $\exists y \in \mathbb{N}\ y > n \wedge Prime(y)$
    - In the first case, N itself was instantiation of $\exists y$, and in the second case, k was.
  - There are no more cases, so we showed that $\exists y \in \mathbb{N}\ y > n \wedge Prime(y)$, contradicting the assumption for an arbitrary n
  - Therefore, by universal generalization, the assumption is false, so $\forall x \in \mathbb{N}\ Prime(x) \rightarrow \exists y \in \mathbb{N}\ y > x \wedge Prime(y)$

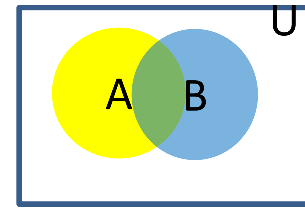
□ (Done).

Puzzle: the barber

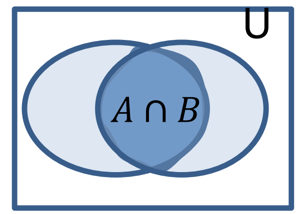
- In a certain village, there is a (male) barber who shaves all and only those men of the village who do not shave themselves.

- *Question: who shaves the barber?*

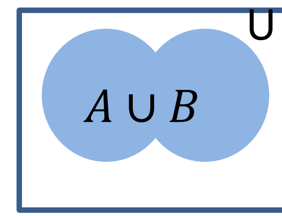
Venn diagrams of set operations



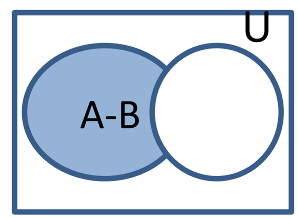
- Let A and B be two sets.
  - Such as A={1,2,3} and B={ 2,3,4}
- **Intersection** $A \cap B = \{\, x \mid x \in A \wedge x \in B\}$
  - The green part of the picture above
  - $A \cap B = \{2,3\}$



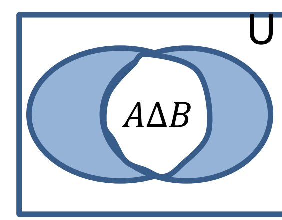
- **Union** $A \cup B = \{\, x \mid x \in A \vee x \in B\}$
  - The coloured part in the top picture.
  - $A \cup B = \{1,2,3,4\}$



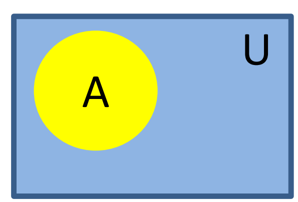
- **Difference** $A - B = \{x \mid x \in A \wedge x \notin B\}$
  - The yellow part in the top picture.
  - $\mathrm{A} - B = \{1\}$



- **Symmetric difference** $A \,\Delta\, B = (A - B) \cup (B - A)$
  - The yellow and blue parts of the top picture.
  - $\mathrm{A}\Delta B = \{1,4\}$

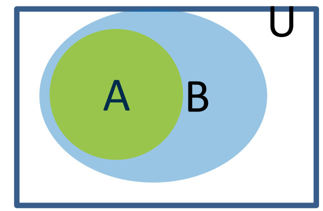


- **Complement** $\overline{A} = \{x \in U \mid x \notin A\}$
  - The blue part on the bottom Venn diagram
  - If universe U = ℕ, $\overline{A} = \{x \in \mathbb{N} \mid x \notin \{1,2,3\}\ \}$

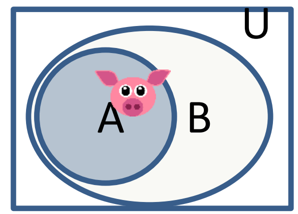


Subsets and operations

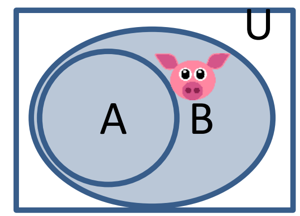
- If $A \subseteq B$ then



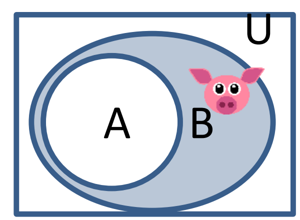
  - Intersection $A \cap B =$
    - A



  - Union $A \cup B =$
    - B



  - Difference $A - B =$
    - $\emptyset$
  - Difference B – A =
    - $\overline{A} - \overline{B}$



Size (cardinality)

- If a set A has n elements, for a natural number n, then A is a **finite** set and its **cardinality** is |A|=n.
  - $|\{1,2,3\}| = 3$
  - $|\emptyset| = 0$

- Sets that are not finite are **infinite**. More on cardinality of infinite sets in a couple of lectures…
  - $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$
  - $\mathbb{R}, \mathbb{C}$
  - $\{0,1\}^*$: set of all finite-length binary strings.

Enrollment puzzle

- There are 160 students in 1000 this semester
- There are 105 students in 1002
- The total number of students in either of these two courses is 200
- How many students are in both 1000 and 1002?