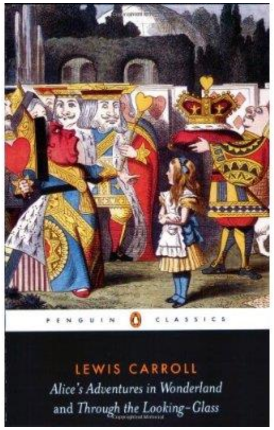
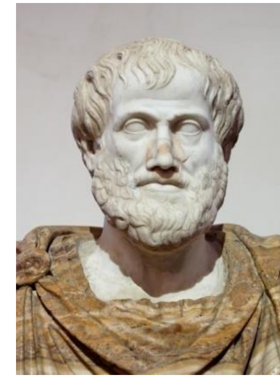




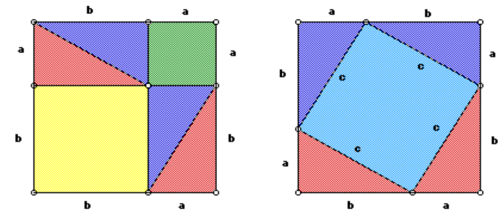
COMP 1002

Intro to Logic for Computer Scientists

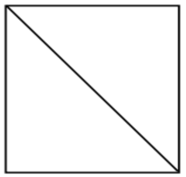
Lecture 15

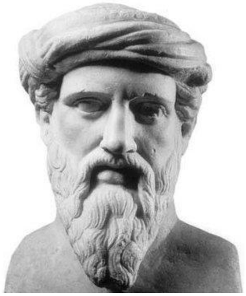
B 5 2 J

Types of proofs



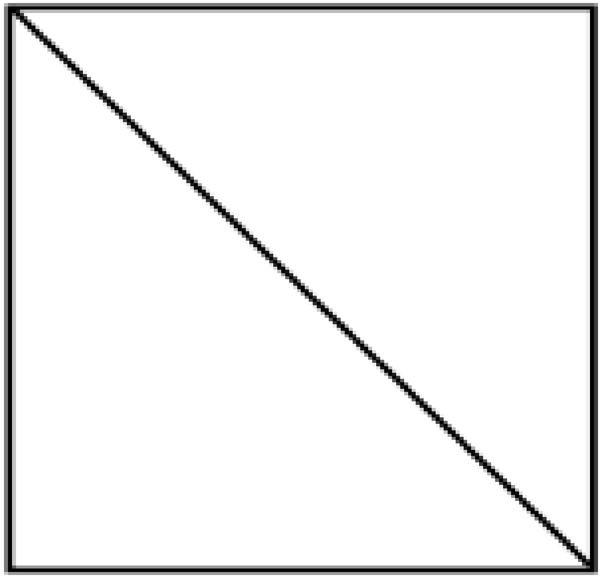
- Direct proof of $\forall x\, F(x)$
  - Show that $F(x)$ holds for arbitrary x, then use universal generalization.
    - Often, $F(x)$ is of the form $G(x) \rightarrow H(x)$
  - Example: A sum of two even numbers is even.
- Proof by contraposition
  - To prove $\forall x \;\; G(x) \rightarrow H(x)$, prove $\forall x\, \neg H(x) \rightarrow \neg G(x)$
  - Example: If square of an integer is even, then this integer is even.
- Proof by cases
  - If can write $\forall x\, F(x)$ as $\forall x (G_1(x) \vee G_2(x)) \rightarrow H(x)$, prove $\forall\, x\, \big(G_1(x) \rightarrow H(x)\big) \wedge (G_2(x) \rightarrow H(x)))$
  - Example: triangle inequality $(|x+y| \leq |x| + |y|)$
- Proof by contradiction
  - To prove $\forall x\, F(x)$, prove $\forall x\, \neg F(x) \rightarrow FALSE$
  - Example: $\sqrt{2}$ is not a rational number.
  - Example: There are infinitely many primes.

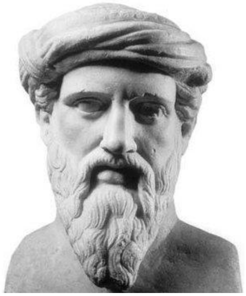


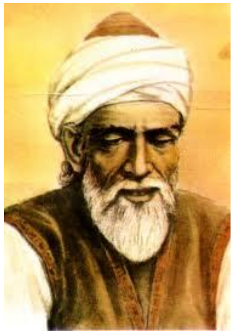


Square root of 2

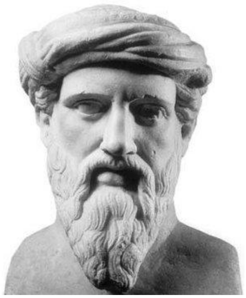
- Is it possible to have a Pythagorean triple with a=b=1?
- Not quite: $1^2 + 1^2 = 2,$ so the third side would have to be $\sqrt{2}$.
- Is it at least possible to represent $\sqrt{2}$ as a ratio of two integers?...
  - Pythagoras and others tried...



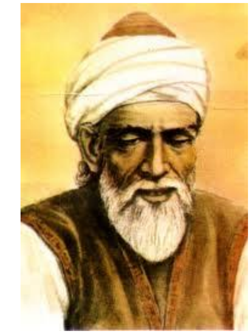




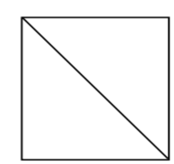
– What are Rational and irrational numbers?

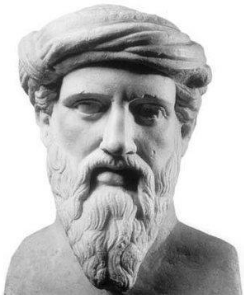


Rational and irrational numbers



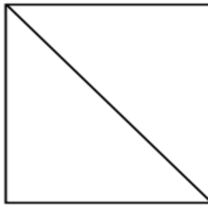
- The numbers that are representable as a fraction of two integers are **rational** numbers. Set of all rational numbers is $\mathbb{Q}$.
- Numbers that are not rational are **irrational**.
  - Pythagoras figured out that the diagonal of a square is not comparable to the sides, but did not think of it as a number.
    - More like something weird.
  - It seems that irrational numbers started being treated as numbers in 9^{th} century in the Middle East.
    - Starting with a Persian mathematician and astronomer Abu-Abdullah Muhammad ibn Īsa Māhānī (Al-Mahani).
- Rational and irrational numbers together form the set of all real numbers.
  - Any sequence of digits, potentially infinite after a decimal point, is a real number. Any point on a line.
- Irrationality of $\sqrt{2}$ is a classic proof by contradiction.

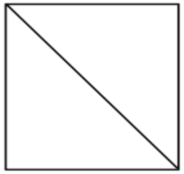


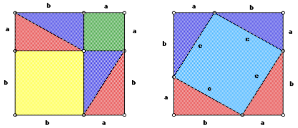


Definition of rational

- We need a slightly more precise definition of rational numbers for our proof that $\sqrt{2}$ is irrational.
- *Definition* (of rational and irrational numbers):
  - A real number r is **rational** iff $\exists m, n \in \mathbb{Z}, n \neq 0 \wedge \gcd(m, n) = 1 \wedge r = \frac{m}{n}$.
    - Reminder: **greatest common divisor gcd(m,n)** is the largest integer which divides both m and n. When d=1, m and n are **relatively prime**.
    - Any fraction can be simplified until the numerator and denominator are relatively prime, so it is not a restriction
  - A real number which is not rational is called **irrational**.



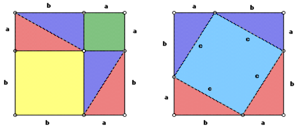




Proof by contradiction

- *Definition* (of rational and irrational numbers):
  - A real number r is **rational** iff $\exists m, n \in \mathbb{Z}, n \neq 0 \wedge \gcd(m, n) = 1 \wedge r = \frac{m}{n}$.

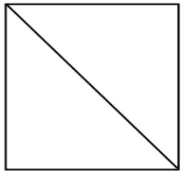
- *Theorem*: Square root of 2 is irrational.

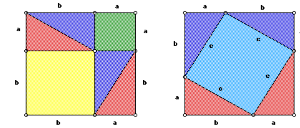


Proof by contradiction

- To prove $\forall x \; F(x)$, prove $\forall x \neg F(x) \rightarrow FALSE$
  - Universal instantiation: “let n be an arbitrary element of the domain S of $\forall x$”
  - Suppose that $\neg F(n)$ is true.
  - Derive a contradiction.
  - Conclude that $F(n)$ is true.
  - By universal generalization, $\forall x \, F(x)$ is true.

☐ (Done).

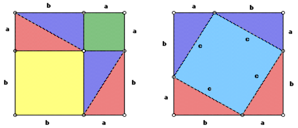




Proof by contradiction

- *Theorem*: Square root of 2 is irrational.
- *Proof*:
  - Suppose, for the sake of contradiction, that $\sqrt{2}$ is rational. Then there exist relatively prime m, n ∈ $\mathbb{Z}$, $n \neq 0$ such that $\sqrt{2} = \frac{m}{n}$.
  - By algebra, squaring both sides we get $2 = \frac{m^2}{n^2}$.
  - Thus m^2 is even, and by the theorem we just proved, then m is even. So $m = 2k$ for some k.
  - $2n^2 = 4\,k^2$, so $n^2 = 2k^2$, and by the same argument n is even.
  - This contradicts our assumption that m and n are relatively prime. Therefore, such m and n cannot exist, and so $\sqrt{2}$ is not rational.

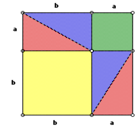
□ (Done).

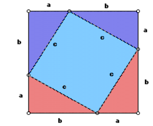


Proof by cases

- Use the tautology $(p_1 \vee p_2) \wedge (p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \rightarrow q$
- If $\forall x\, F(x)$ is $\forall x(G_1(x) \vee G_2(x)) \rightarrow H(x)$,
- prove $\big(G_1(x) \rightarrow H(x)\big) \wedge (G_2(x) \rightarrow H(x))$.
- Proof:
  - Universal instantiation: “let n be an arbitrary element of the domain S of $\forall x$ ”
  - Case 1: $G_1(n) \rightarrow H(n)$
  - Case 2: $G_2(n) \rightarrow H(n)$
  - Therefore, $(G_1(n) \vee G_2(n)) \rightarrow H(n)$,
  - Now use universal generalization to conclude that $\forall x\, F(x)$ is true.
- This generalizes for any number of cases $k \geq 2$.

□ (Done).

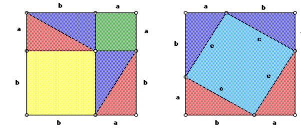




Proof by cases.

- *Definition* (of odd integers):
  - An integer n is **odd** iff $\exists k \in \mathbb{Z}, n = 2 \cdot k + 1$.
- *Theorem*: Sum of an integer with a consecutive integer is odd.
  - $\forall x \in \mathbb{Z} \; Odd(x + (x + 1))$.
- *Proof*:
  - Suppose n is an arbitrary integer.
  - Case 1: n is even.
    - So n=2k for some k (by definition).
    - Its consecutive integer is n+1 = 2k+1. Their sum is (n+(n+1))= 2k + (2k+1) = 4k+1. (axioms).
    - Let $l = 2k$. Then $4k + 1 = 2l + 1$ is an odd number (by definition). So in this case, n+(n+1) is odd.
  - Case 2: n is odd.
    - So n=2k+1 for some k (by definition).
    - Its consecutive integer is n+1 = 2k+2. Their sum is (n+(n+1))= (2k+1) + (2k+2) = 2(2k+1)+1. (axioms).
    - Let $l = 2k + 1$. Then n+(n+1) = 2(2k+1)+1= $2l + 1$, which is an odd number (by definition). So in this case, n+(n+1) is also odd.
  - Since in both cases n+(n+1) is odd, it is odd without additional assumptions. Therefore, by universal generalization, get $\forall x \in \mathbb{Z} \; Odd(x + (x + 1))$.

□ (Done).



Proof by cases

- *Definition*: an absolute value of a real number r is a non-negative real number |r| such that if $|r| = r$ if $r \geq 0$, and $|r| = -r$ if $r < 0$
  - Claim 1: $\forall x \in \mathbb{R}, |-x| = |x|$
  - Claim 2: $\forall x \in \mathbb{R}, -|x| \leq x \leq |x|$
- *Theorem*: for any two reals, sum of their absolute values is at least the absolute value of their sum.
  - $\forall x, y \in \mathbb{R}\ \ |x + y| \leq |x| + |y|$
- *Proof*:
  - Let r and s be arbitrary reals. (universal instantiation)
  - Case 1: Let $r + s \geq 0$.
    - Then $|r + s| = r + s$ (definition of ||)
    - Since $r \leq |r|$ and $s \leq |s|$ (claim 2), r+$s \leq |r| + |s|$ (axioms),
    - so $|r + s| =$ r+$s \leq |r| + |s|$, which is what we need.
  - Case 2: Let $r + s < 0$.
    - Then $|r + s| = -(r + s) = (-r) + (-s)$ (definition of ||)
    - Since $-r \leq |-r| = |r|$ and $-s \leq |-s| \leq |s|$ (claims 1 and 2),
    - |r+$s| = (-r) + (-s) \leq |r| + |s|$ (axioms), which is what we need.
  - Since in both cases |r+$s| \leq |r| + |s|$, and there are no more cases, |r+$s| \leq |r| + |s|$ without additional assumptions. By universal generalization , can now get $\forall x, y \in \mathbb{R}\ \ |x + y| \leq |x| + |y|$.

□ (Done).

Puzzle: the barber

- In a certain village, there is a (male) barber who shaves all and only those men of the village who do not shave themselves.

- *Question: who shaves the barber?*