COMP 1002

Intro to Logic for Computer Scientists

Lecture 11
Puzzle 10

• The first formulation of the famous liar’s paradox, attributed to a Cretan philosopher Epimenides, stated “All Cretans are liars”.

Is this really a paradox?
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• The first formulation of the famous liar’s paradox, attributed to a Cretan philosopher Epimenides, stated

“All Cretans are liars”.

Is this really a paradox?

– The negation of “all” is “exists”,
  • just like the negation of “and” is “or”
– So if Epimenides lied, what is true is that there are some truth-tellers on Crete (and potentially some liars, too)
– And Epimenides is one of the liars.
– However, “I am lying” would be a paradox.
“NOT” makes life harder

• It is easy to visualize a tree, a number, or a person. It is harder to visualize a “not a tree”, “not a number” or “not a person”

• So “NOT (ALL trees have leaves)” is harder to understand than “some trees have something other than leaves (e.g., needles).

• Here we really need to pay attention to the domain of quantifiers! It stays the same when negating.
  – Not all integers are even: \( \neg(\forall x \in \mathbb{Z} \text{ Even}(x)) \)
  \( \equiv \)
  – Some integers are not even \( \exists x \in \mathbb{Z} \neg \text{Even}(x) \)
Mixing quantifiers

- We can make statements of predicate logic mixing existential and universal quantifiers.
- Order of variables under the same quantifier does not matter. Under different ones does.
  - Predicate: Loves(x,y). Domain: people.
    - Everybody loves somebody: $\forall x \exists y \text{ Loves}(x,y)$
      - Normal people
    - Somebody loves everybody: $\exists x \forall y \text{ Loves}(x,y)$
      - Mother Teresa
    - Everybody is loved by somebody $\forall x \exists y \text{ Loves}(y,x)$
      - Their mother
    - Somebody is loved by everybody $\exists x \forall y \text{ Loves}(y,x)$
      - Elvis Presley
    - Everybody is loved by everybody $\forall x \forall y \text{ Loves}(x,y)$
      - Domain is a good family (not Meow-stery family)
Negating mixed quantifiers

• Now, a “not” in front of such a sentence means all $\forall$ and $\exists$ are interchanged, and the inner part becomes negated.

  – Everybody loves somebody: $\forall x \ \exists y \ \text{Loves}(x, y)$
    • Somebody does not love anybody $\exists x \ \forall y \ \neg \text{Loves}(x, y)$
    • Can also say “Somebody loves nobody” in English.
    • Not the same as “somebody does not love everybody”: here, “somebody does not (love everybody)” meaning $\exists x \ \neg (\forall y \ \text{Loves}(x, y)) \equiv \ \exists x \ \exists y \ \neg \text{Loves}(x, y)$
    • But the formula $\exists x \ \exists y \ \neg \text{Loves}(x, y)$ is the negation of $\forall x \ \forall y \ \text{Loves}(x, y)$
Negating mixed quantifiers

- Everybody loves somebody: $\forall x \exists y \text{ Loves}(x,y)$
  - Somebody does not love anybody $\exists x \forall y \neg \text{ Loves}(x,y)$

- Somebody loves everybody: $\exists x \forall y \text{ Loves}(x,y)$
  - Everyone doesn’t like somebody $\forall x \exists y \neg \text{ Loves}(x,y)$

- Everybody is loved by somebody $\forall x \exists y \text{ Loves}(y,x)$
  - Somebody is not loved by anybody $\exists x \forall y \neg \text{ Loves}(y,x)$

- Somebody is loved by everybody $\exists x \forall y \text{ Loves}(y,x)$
  - For everyone, somebody does not love them $\forall x \exists y \neg \text{ Loves}(y,x)$

- Everybody is loved by everybody $\forall x \forall y \text{ Loves}(y,x)$
  - Somebody does not love someone $\exists x \exists y \neg \text{ Loves}(y,x)$
Scope of quantifiers

• Like in programming, a scope of a quantified variable continues until a new variable with the same name is introduced.
  - \( \forall x (\exists y P(x, y)) \land (\exists y Q(x, y)) \)
    - For everybody there is somebody who loves them and somebody who hates them.
  - Not the same as \( \forall x (\exists y P(x, y) \land Q(x, y)) \)
    - For everybody there is somebody who both loves and hates them.

• Better to avoid using same names for different variables – it is confusing.
  - \( \forall x (\exists y P(x, y)) \land (\exists y Q(x, y)) \)
    \[ \equiv \]
  - \( \forall x (\exists y P(x, y)) \land (\exists z Q(x, z)) \)
    \[ \equiv \]
  - \( \forall x \exists y \exists z P(x, y) \land Q(x, z) \)
    \[ \equiv \]
  - \( \forall x \exists z \exists y P(x, y) \land Q(x, z) \)
Equivalence for predicate logic

• Two predicate logic formulas are equivalent if they have the same truth value for every setting of free variables, no matter what the predicates and their universes are.

  – \((\exists y \ P(x, y)) \land (\exists y \ Q(x, y))\)
    \[\equiv\]
  – \((\exists y \ P(x, y)) \land (\exists z \ Q(x, z))\)
    \[\equiv\]
  – \(\exists y \ \exists z \ P(x, y) \land Q(x, z)\)
    \[\equiv\]
  – \(\exists z \ \exists y \ P(x, y) \land Q(x, z)\)

  – But \(\exists x \ \forall y \ P(x, y, z)\) is not equivalent to \(\forall y \ \exists x \ P(x, y, z)\)
Prenex normal form

• When all quantified variables have different names, can move all quantifiers to the front of the formula, and get an equivalent formula: this is called **prenex normal form**.
  - $\forall x \exists y \exists z \ P(x, y) \land Q(x, z)$ is in prenex normal form
  - $\forall x (\exists y P(x, y)) \land (\exists z Q(x, z))$ is not in prenex normal form.

• Be careful with implications: when in doubt, open into $\neg A \lor B$. Move all negations inside.
  - $\forall x ((\exists y P(x, y)) \rightarrow Q(x))$ actually has two universal quantifiers!
  - Its equivalent in prenex normal form is $\forall x \forall y (\neg P(x, y) \lor Q(x))$
Quantifiers and conditionals

- Which statements are true?
  - All squares are white. All white shapes are squares.
  - All circles are blue. All blue shapes are circles.
  - All lemurs live in the trees. All animals living in the trees are lemurs.
  - $\forall x \in S, \ P(x) \rightarrow Q(x)$
    - For all objects, if it is white, then it is a square.
    - If an object is white, then it is a square.
    - If an animal is a lemur, then it lives in the trees.
• Then you should say what you mean,’ the March Hare went on.

• ‘I do,’ Alice hastily replied; ‘at least—at least I mean what I say—that’s the same thing, you know.’

• ‘Not the same thing a bit!’ said the Hatter. ‘You might just as well say that “I see what I eat” is the same thing as “I eat what I see”!’

• ‘You might just as well say,’ added the March Hare, ‘that “I like what I get” is the same thing as “I get what I like”!’

• ‘You might just as well say,’ added the Dormouse, who seemed to be talking in his sleep, ‘that “I breathe when I sleep” is the same thing as “I sleep when I breathe”!’

“Alice’s Adventures in Wonderland”
by Lewis Carroll