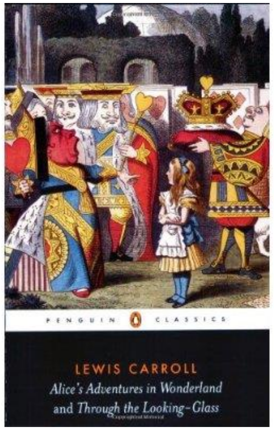
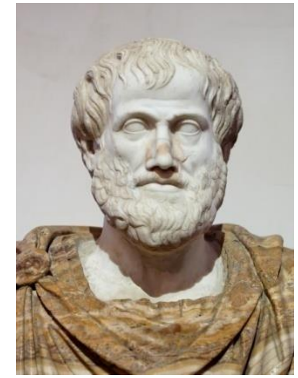


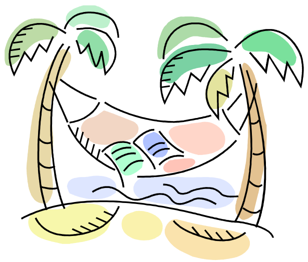


COMP 1002

Intro to Logic for Computer Scientists

Lecture 11

B 5 2 J

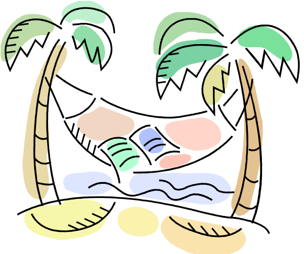


Puzzle 10

- The first formulation of the famous liar's paradox, attributed to a Cretan philosopher Epimenides, stated

“All Cretans are liars”.

Is this really a paradox?



Puzzle 10

- The first formulation of the famous liar's paradox, attributed to a Cretan philosopher Epimenides, stated

"All Cretans are liars".

Is this really a paradox?

- *The negation of "all" is "exists",*
  - *just like the negation of "and" is "or"*
- *So if Epimenides lied, what is true is that there are **some** truth-tellers on Crete (and potentially some liars, too)*
- *And Epimenides is one of the liars.*
- *However, "I am lying" would be a paradox.*

"NOT" makes life harder

- It is easy to visualize a tree, a number, or a person. It is harder to visualize a "not a tree", "not a number" or "not a person"
- So "NOT (ALL trees have leaves)" is harder to understand than "some trees have something other than leaves (e.g., needles).

- Here we really need to pay attention to the domain of quantifiers! It stays the same when negating.
  - Not all integers are even: $\neg(\forall x \in \mathbb{Z}\ Even(x))$

    $$\equiv$$

  - Some integers are not even $\exists x \in \mathbb{Z}\ \neg Even(x)$

Mixing quantifiers

- We can make statements of predicate logic mixing existential and universal quantifiers.
- Order of variables under the same quantifier does not matter. Under different ones does.
  - Predicate: Loves(x,y). Domain: people.
  - Everybody loves somebody: ∀x ∃y Loves(x,y)
    - Normal people

  - Somebody loves everybody: ∃x ∀y Loves(x,y)
    - Mother Teresa

  - Everybody is loved by somebody ∀x ∃y Loves(y,x)
    - Their mother

  - Somebody is loved by everybody ∃x ∀y Loves(y,x)
    - Elvis Presley

  - Everybody is loved by everybody ∀x ∀y Loves(x,y)
    - Domain is a good family (not Meow-stery family)

Negating mixed quantifiers

- Now, a “not” in front of such a sentence means all ∀ and ∃ are interchanged, and the inner part becomes negated.

  - Everybody loves somebody: ∀x ∃y Loves(x,y)
    - Somebody does not love anybody ∃ x ∀ y ¬Loves(x,y)
    - Can also say “Somebody loves nobody” in English.
    - Not the same as “somebody does not love everybody”: here, “somebody does not (love everybody)” meaning $\exists x \neg (\forall y\ Loves(x, y)) \equiv \exists x\, \exists y\, \neg Loves(x, y)$
    - But the formula $\exists x\, \exists y\, \neg Loves(x, y)$ is the negation of $\forall x\, \forall y\ Loves(x, y)$

Negating mixed quantifiers

- Everybody loves somebody: $\forall x\ \exists y$ Loves(x,y)
  - Somebody does not love anybody $\exists X\ \forall Y\ \neg$Loves(x,y)

- Somebody loves everybody: $\exists x\ \forall y$ Loves(x,y)
  - Everyone doesn't like somebody $\forall x\ \exists y\ \neg$ Loves(x,y)

- Everybody is loved by somebody $\forall x\ \exists y$ Loves(y,x)
  - Somebody is not loved by anybody $\exists x\ \forall y\ \neg$ Loves(y,x)

- Somebody is loved by everybody $\exists x\ \forall y$ Loves(y,x)
  - For everyone, somebody does not love them $\forall x\ \exists y\ \neg$ Loves(y,x)
- Everybody is loved by everybody $\forall x\ \forall y$ Loves(y,x)
  - Somebody does not love someone $\exists x\ \exists y\ \neg$ Loves(y,x)

Scope of quantifiers

- Like in programming, a scope of a quantified variable continues until a new variable with the same name is introduced.
  - $\forall x\ \big(\exists y\ \ P(x, y\ )\big) \land \big(\exists y\ Q(x, y)\big)$
    - For everybody there is somebody who loves them and somebody who hates them.
  - Not the same as $\forall x\ \big(\exists y\ \ P(x, y\ ) \land Q(x, y)\big)$
    - For everybody there is somebody who both loves and hates them.

- Better to avoid using same names for different variables – it is confusing.
  - $\forall x\ \big(\exists y\ \ P(x, y\ )\big) \land \big(\exists y\ Q(x, y)\big)$

    $\equiv$
  - $\forall x\ \big(\exists\ y\ P(x, y)\big) \land \big(\exists z\ Q(x, z)\big)$

    $\equiv$
  - $\forall x \exists\ y\ \exists z\ P(x, y) \land Q(x, z)$

    $\equiv$
  - $\forall x \exists\ z\ \exists y\ P(x, y) \land Q(x, z)$

Equivalence for predicate logic

- Two predicate logic formulas are equivalent if they have the same truth value for every setting of free variables, no matter what the predicates and their universes are.
  - $\left(\exists y\ P(x,y)\right) \wedge \left(\exists y\ Q(x,y)\right)$

    $\equiv$
  - $\left(\exists\ y\ P(x,y)\right) \wedge \left(\exists z\ Q(x,z)\right)$

    $\equiv$
  - $\exists\ y\ \exists z\ P(x,y) \wedge Q(x,z)$

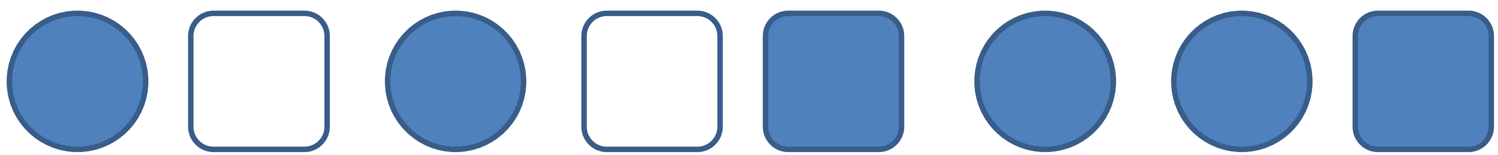
    $\equiv$
  - $\exists\ z\ \exists y\ P(x,y) \wedge Q(x,z)$
  - But $\exists x\ \forall y\ P(x,y,z)$ is not equivalent to $\forall y\ \exists x\ P(x,y,z)$

Prenex normal form

- When all quantified variables have different names, can move all quantifiers to the front of the formula, and get an equivalent formula: this is called **prenex normal form**.
  - $\forall x \exists\, y\, \exists z\, P(x, y) \wedge Q(x, z)$ is in prenex normal form
  - $\forall x \left(\exists\, y\, P(x, y)\right) \wedge \left(\exists z\, Q(x, z)\right)$ is not in prenex normal form.
- Be careful with implications: when in doubt, open into $\neg A \vee B$. Move all negations inside.
  - $\forall x\ \left(\left(\exists y\, P(x, y)\right) \rightarrow Q(x)\ \right)$ actually has two universal quantifiers!
  - Its equivalent in prenex normal form is $\forall x\, \forall y \left(\neg P(x, y) \vee Q(x)\right)$

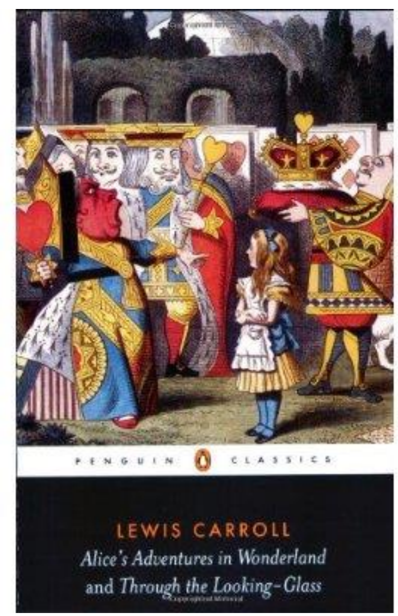
Quantifiers and conditionals

- Which statements are true?
  - All squares are white. All white shapes are squares
  - All circles are blue. All blue shapes are circles.



  - All lemurs live in the trees. All animals living in the trees are lemurs.
  - $\forall x \in S,\ P(x) \rightarrow Q(x)$
    - For all objects, if it is white, then it is a square.
    - If an object is white, then it is a square.
    - If an animal is a lemur, then it lives in the trees.

- Then you should say what you mean,’ the March Hare went on.
- `I do,’ Alice hastily replied; `at least–at least I mean what I say–that’s the same thing, you know.’
- `Not the same thing a bit!’ said the Hatter. `You might just as well say that “I see what I eat” is the same thing as “I eat what I see”!’
- `You might just as well say,’ added the March Hare, `that “I like what I get” is the same thing as “I get what I like”!’
- `You might just as well say,’ added the Dormouse, who seemed to be talking in his sleep, `that “I breathe when I sleep” is the same thing as “I sleep when I breathe”!’



“Alice’s Adventures in Wonderland”
by Lewis Carroll