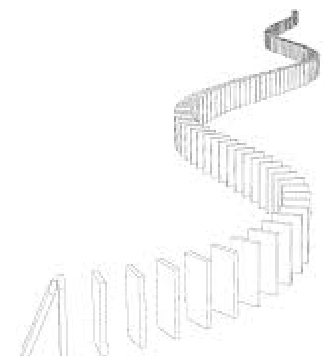
COMP 1002

Logic for Computer Scientists

Lecture 29



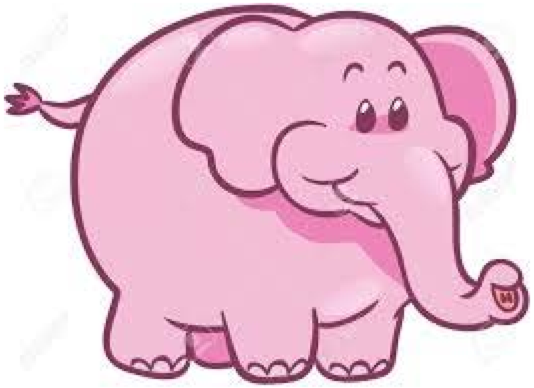
Admin stuff

- Assignment 5 is posted.
  - Due April 2nd.

- Next week:
  - "Mini-lab" on Monday, April 3rd.
  - Review for the final exam on Tuesday, April 4th.
  - Practice final exam on Wednesday, April 5th
    - 9am-11am in CS-1019.

Probabilities and pink elephants

- What is the probability that walking down George street you'd see a pink elephant?
  - Your friend says: "It is ½! You will either see the pink elephant, or not!"
    - Do you agree?

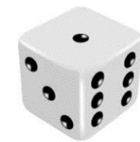


Probabilities and distributions

- What if outcomes are not equally likely?
  - Biased coins, etc.
- A function $Pr: S \to \mathbb{R}$ is a **probability distribution** on (a finite set) S if Pr satisfies the following:
  - For any outcome s $\in S$, $0 \leq \Pr(s) \leq 1$
  - $\Sigma_{\{s \in S\}} \Pr(s) = 1$
- **Uniform distribution**: for all $s \in S, \Pr(s) = 1/|S|$
  - all outcomes are equally likely
  - Fair coin: $\Pr(heads) = \Pr(tails) = \frac{1}{2}$
- Biased coin: say heads twice as likely as tails.
  - $\Pr(heads) + \Pr(tails) = 1. \Pr(heads) = 2 * \Pr(tails)$
  - So $\Pr(heads) = \frac{2}{3}, \Pr(tails) = \frac{1}{3}$

Probabilities of events

- Probability of an event A is a sum of probabilities of the outcomes in A:
  - Pr(A)=$\Sigma_{\{a \in A\}} \Pr(a)$



- Probability of A not occurring:
  - $\Pr(\bar{A}) = 1 - \Pr(A)$
- Probability of the union of two events (either A or B happens) is

$$\Pr(A \cup \mathrm{B}) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

    - By principle of inclusion-exclusion
  - If A and B are disjoint, then $\Pr(A \cap B) = 0,$ so $\Pr(A \cup B) = \Pr(A) + \Pr(B)$
- In general, if events $A_1 \dots A_n$ are pairwise disjoint
    - that is, $\forall i, j$ if $i \neq j$ then $A_i \cap A_j = \emptyset$
  - Then Pr($\cup_{i=1}^{n} \ A_i) = \Pr(A_1 \cup A_2 \cup \cdots \cup A_n) = \Sigma_{i=1}^{n} \ \Pr(A_i)$
    - That is, probability of that any of the events happens is the sum of their individual probabilities.

Probabilities of events

- Suppose a die is biased so that 3 appears twice as often as any other number (others equally likely).
  - Probability of 3: 2/7. Probabilities of others: 1/7

- What is the probability that an odd number appears?
  - Event: A={1,3,5}
  - Pr(A) = 1/7+2/7+1/7=4/7.

- What is a probability that either an odd number or a number divisible by 3 appears?
  - A={1,3,5}. B = {3,6} . $A \cap B = \{3\}$
  - Pr(A) = 4/7. Pr(B) = 3/7. $\Pr(A \cap B) = 2/7$
  - $\Pr(A \cup B) = \Pr(\{1,3,5,6\})$

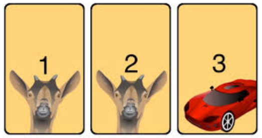
$$= \frac{4}{7} + \frac{3}{7} - \frac{2}{7} = \frac{1}{7} + \frac{2}{7} + \frac{1}{7} + \frac{1}{7} = \frac{5}{7}$$

Birthday paradox

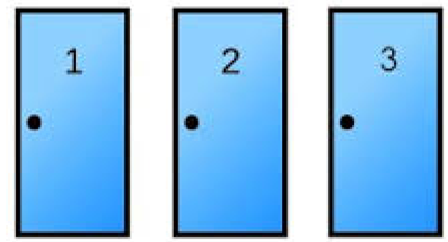
- How many people have to be in the room so that probability that two of them have the same birthday is at least ½?

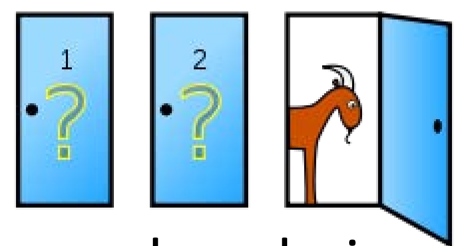
Birthday paradox

- How many people have to be in the room so that probability that two of them have the same birthday is at least ½?
  - Considering all birthdays independent: no twins!
  - And considering all days equally likely
    - Otherwise probability would be higher.
  - Even counting leap years: 366 days.
- Product rule: number of combinations of birthdays of the first i people is P(i, 366) = 366*365*…*(366-i+1)
  - Probability that the first i people all have different birthday is $\frac{P(i,366)}{366^i} = \frac{365}{366}\frac{364}{366}\ldots\frac{(366-i+1)}{366}$
  - So with probability $1 - \frac{P(i,366)}{366^i}$ at least two out of first i people have birthday on the same day.
  - That comes up to about i =23 people to reach ½.

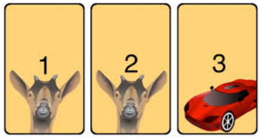


Puzzle: Monty Hall problem





- Let’s make a deal!
  - A player picks a door.
  - Behind one door is a car.
  - Behind two others are goats.
- A player chooses a door.
  - A host opens another door
  - Shows a goat behind it.
  - And asks the player if she wants to change her choice.
- Should she switch?
  - Originally, probability of picking the car is 1/3
  - If she first picked a door with a car: (1/3 probability)
    - Then she would switch to a goat.
  - If she first picked a door with a goat (2/3 probability)
    - Then she would switch to a car.

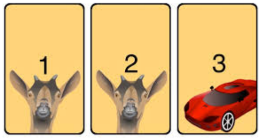


Conditional probabilities

- **Conditional probability** of an event A given event B, denoted $\Pr(A|B)$, is the probability of A if we know that B occurred.
  - Probability of car a behind door 2 if we chose door 1, and door 3 had a goat behind it.
- So it is probability of both A and B, given that we know B happened for sure:

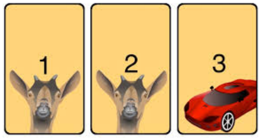
$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

  - Assume that $\Pr(B) > 0$: after all, B did happen.



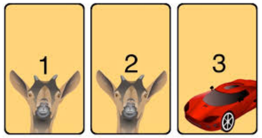
Independent events

- If knowing B gives us no information about A and vice versa, then A and B are **independent** events:
  - Then Pr(A) = Pr(A|B) = $\frac{\Pr(A \cap B)}{\Pr(B)}$.
  - So A and B are independent iff $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$.
    - In general, events $A_1 \ldots A_n$ can be **pairwise independent** (that is, any two A_i, A_j) are independent, or (stronger condition) **mutually independent**: $\Pr(\cap_{i=1}^{n} A_i) = \Pi_{i=1}^{n} \Pr(A_i)$
  - For example, different coin tosses/dice rolls are usually considered independent.



Bayes theorem

- Consider a medical test that
  - Has false positive rate of 3% (healthy labeled as sick)
    - Specificity 97%
  - Has false negative rate of 1% (sick labeled as healthy).
    - Sensitivity 99%
- What is the probability that a person has the disease given that the test came positive?
  - Let A: person tested positive, B: person is sick. $\Pr(B|A)$?
  - Pr(A|B) = 0.99, $\Pr(\bar{A}|B) = 0.01\ldots$
- Not enough information!

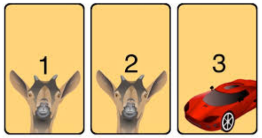


Bayes theorem

- **Bayes theorem** allows us to get $\Pr(B|A)$ from $\Pr(A|B)$, if we know probabilities of A and B:

$$\Pr(B|A) = \frac{\Pr(\mathrm{A}|\mathrm{B})\cdot\Pr(B)}{\Pr(A)} = \frac{\Pr(\mathrm{A}|\mathrm{B})\cdot\Pr(B)}{\Pr(A|B)\Pr(B)+\Pr(A|\bar{B})\Pr(\bar{B})}$$

- Proof:
  - $\Pr(A|B) = \frac{\Pr(A\cap B)}{\Pr(B)}$. $\Pr(B|A) = \frac{\Pr(A\cap B)}{\Pr(A)}$.
  - $\Pr(\mathrm{A} \cap B) = \Pr(A|B)\Pr(B) = \Pr(B|A)\Pr(A)$
  - So $\Pr(B|A) = \Pr(A|B)\Pr(B) / \Pr(A)$



Bayes theorem

- Consider a medical test that
  - Has false positive rate of 3% (healthy labeled as sick).
  - Has false negative rate of 1% (sick labeled as healthy).
  - Tests for a disease that occurs in 5 in 1000 people.
- What is the probability that a person has the disease given that the test came positive?
  - Let A: person tested positive, B: person is sick. $\Pr(B|A)$?
  - Pr(A|B) = 0.99, Pr($\bar{A}|B$) $= 0.01$. Pr($\bar{A}|\bar{B}$) = 0.97 , Pr($A|\bar{B}$) $= 0.03$
  - $\Pr(B) = 0.005.$
  - $\Pr(A) = \Pr(A|B)\Pr(B) + \Pr(A|\bar{B})\Pr(\bar{B})$ = 0.0348
  - By Bayes theorem, Pr(B|A) = $\frac{\Pr(A|B)\Pr(B)}{\Pr(A)}$ = 0.1422
- So the probability that a person who tested positive has the disease is just 0.1422, that is, 14.22%.
  - By a similar argument, probability that a person who tested negative does not have a disease is whopping 0.99995 = 99.995%.

Hat-check problem

- Suppose n men came to an event, and checked in their hats at the door.
  - On the way out, in a hurry, they each picked up a random hat.
  - On average, how many men picked their own hat?