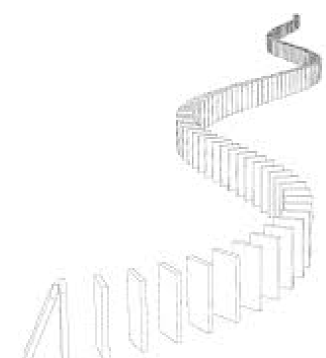
COMP 1002

Logic for Computer Scientists

Lecture 28

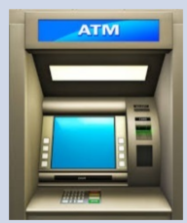
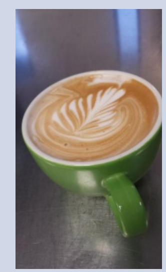
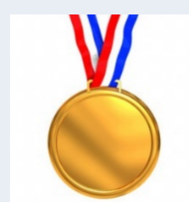


Admin stuff

- Assignment 5 to be posted today.
  - Due April 2nd.

- Assignment 4 is postponed until Saturday morning (7am).

Combinatorics summary

Selecting k out of n objects	Order matters	Order ignored
With repetitions	n^k 	$\binom{k+n-1}{k}$ 
Without repetitions	$P(n,k) = \frac{n!}{(n-k)!}$ 	$\binom{n}{k}$ 

Binomial theorem

- Binomial expansion: open parentheses in $(x+y)^n$
- Open the parentheses in $(x+y)^2$
  - $x^2+2xy+y^2$
- Open parentheses in $(x+y)^3$
  - $x^3+xxy+xyx+yxx+xyy+yxy+yyx+y^3$
    $$= x^3+3x^2y+3xy^2+y^3$$
  - That is, a coefficient in front of x^2y is the number of ways to pick one y (or 2 x) out of 3 positions.
  - Call these coefficients **binomial coefficients**.
- B**inomial theorem**

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$$

- Corollary: $\sum_{k=0}^{n}\binom{n}{k}=2^n$

Pascal’s identity and triangle

- How to compute binomial coefficients?
  - First, note only need to compute them for $0 \leq k \leq \lceil \frac{n}{2} \rceil$, since $\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{k!(n-k)!}$

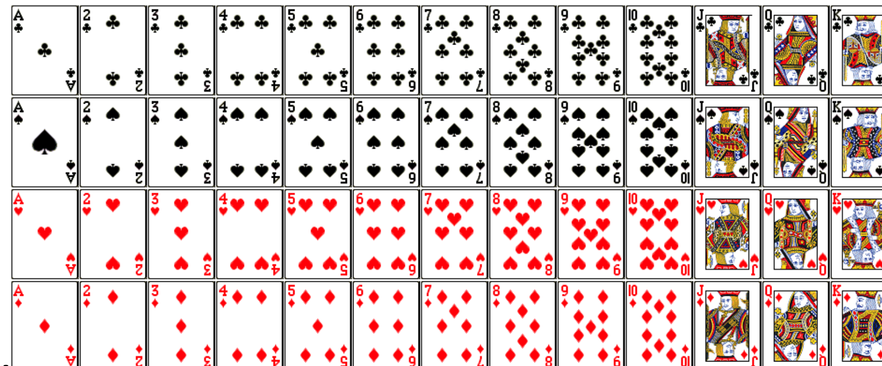
- Pascal’s identity: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$
  - In practice, use Stirling approximation
  - $n! \sim \sqrt{2\pi n}\,(n/e)^n$
  - So $\frac{\mathrm{n}^{\mathrm{k}}}{\mathrm{k}^{\mathrm{k}}} \leq \binom{n}{k} < \frac{(en)^k}{k^k}$
  - And $\ln n! \sim n \ln n - n$

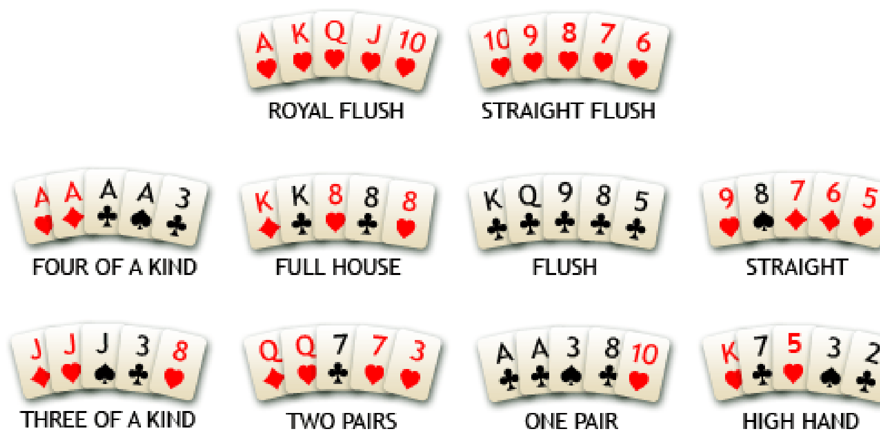
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Pascal’s triangle

Puzzle: playing poker

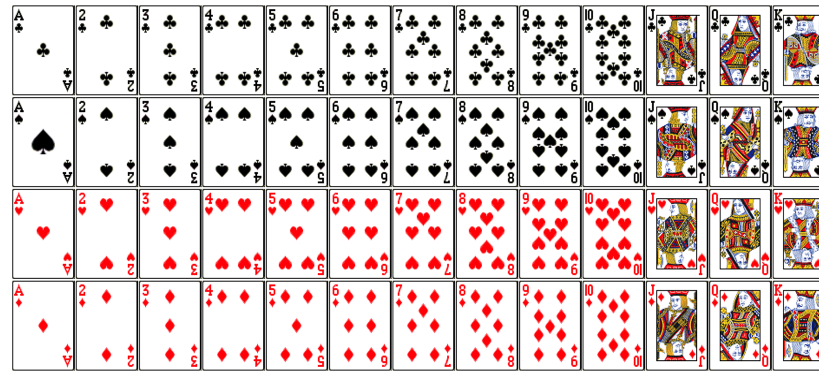
- There are 52 cards in a standard deck; 4 suites of 13 ranks each.
- In poker, some 5-card combinations ("hands") are special:
  - For example, a "three of a kind" consists of three cards with the same rank, together with two cards of other different ranks.
- How many ways are there to choose (ignoring the order)
  - A royal flush?
  - a three of a kind hand?
  - a two pairs hand?
  - other hands?...





Puzzle: playing poker

- How many ways are there to choose (ignoring the order)
  - a royal flush?
    - C(4,1) = 4
  - a three of a kind?
    - pick the rank: 13=C(13,1)
    - Pick 3 out of 4 kinds of this rank: 4=C(4,3)
    - Pick two other ranks: C(12,2)= 66
    - Pick a suite of each of the other ranks: C(4,1)*C(4,1)=16
    - Total: 13*4*66*16=54912

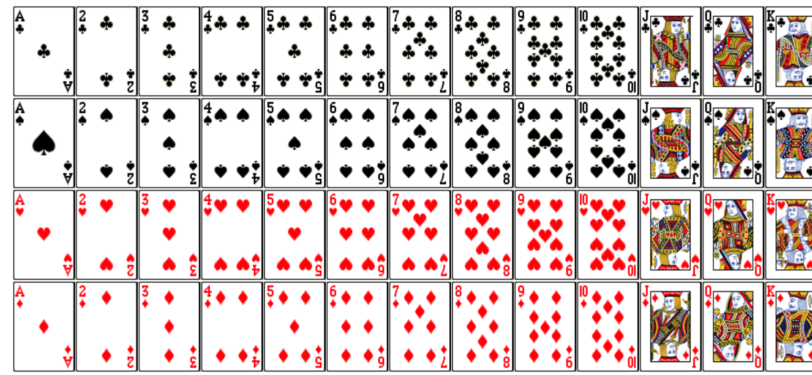


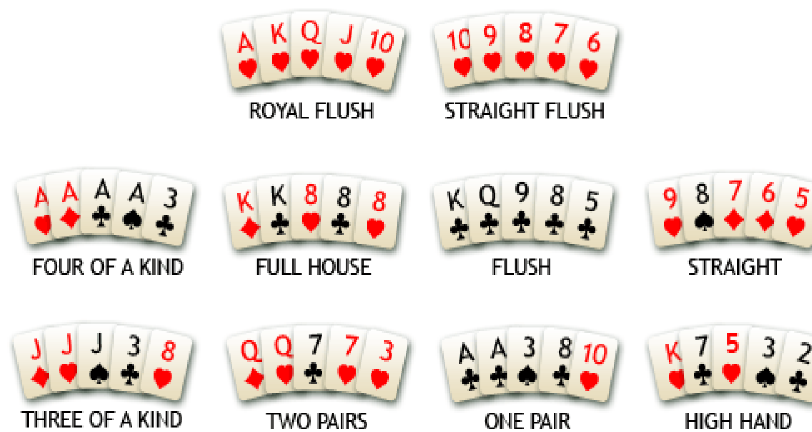
Finite probability

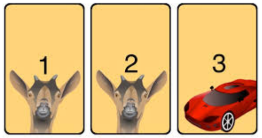
- More common: use the language of probability.
- **Experiments**: producing an **outcome** out of possible choices
  - Tossing a coin: outcome can be "heads"
  - Getting a lottery ticket: outcome can be "win"
- **Sample space S**: set of all possible outcomes.
  - {heads, tails} for coin toss
  - {1,2,3,4,5,6} x {1,2,3,4,5,6} for rolling two dice
- **Event A** $\subseteq S$: subset of outcomes
  - Both dice came up even.
- **Probability** of an event if all outcomes are **equally likely**:
  - Pr(A)= |A|/|S| (fraction of the outcomes that are in the event A).
  - Probability of both dice coming up even:
    - A={ (2,2),(2,4),(4,2),(2,6),(6,2),(4,4), (4,6), (6,4) , (6,6)}. |A| =9, |S|=36
    - P(A)=9/36=1/4
- Can use the same combinatorics we just studied to calculate probabilities (i.e., for finding the size of A).

Puzzle: playing poker

- What is the probability of getting a three of a kind hand?
  - Size of the sample space:
    - C(52, 5) = $\binom{52}{5} =$ 2,598,962
  - Size of the event A:
    - 54,912
  - Probability of A:
    - Pr(A) = $\frac{|A|}{|S|} = 0.0211..$

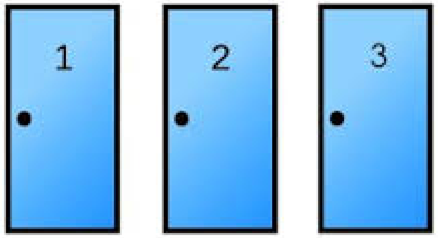




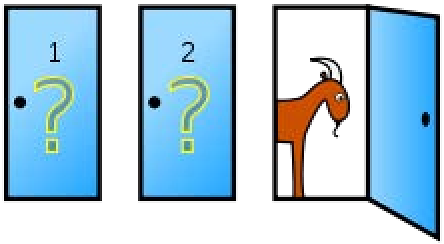


Puzzle: Monty Hall problem

- Let's make a deal!
  - A player picks a door.
  - Behind one door is a car.
  - Behind two others are goats.



- A player chooses a door.
  - A host opens another door
  - Shows a goat behind it.
  - And asks the player if she wants to change her choice.



- Should she switch?