COMP 1002

Logic for Computer Scientists

Lecture 27
Admin stuff

• Assignment 4 is posted.
  – Due March 23rd.
Permutations

• **Permutations**: number of sequences of objects.
  – Without repetition: each object appears once.

• Example: how many ways to assign offices ER-6030 to ER-6033 to Antonina, Dave, Sharene and Yuanzhu?
  – 4 choices to pick who gets ER-6030. This leaves 3 choices to pick who gets ER-6031. Now 2 remain for ER-6032, and the last one is stuck with ER-6033.
    • By the product rule, get 4*3*2*1 = 4!.

• In general, number of permutations of n elements is n!
  – “Permutations”: the difference between choices is only the order of elements.
r-Permutations.

• Before, we talked about permutations of all objects in the set. An \( r \)-permutation \( P(n,r) \) involves taking only \( r \) out of \( n \) objects, and counting the number of possible sequences.
  – That is, the task consists of 1) picking \( r \) out of \( n \) objects. 2) counting number of sequences out of them.

• Example:
  – How many ways to assign ER offices to 4 out of 25 faculty members?
    • Pick one out of 25 to be in ER-6030. One out of remaining 24 to be in ER-6031, one out of 23 for ER-6032, one out of 22 for ER-6033. By the product rule, get \( P(25,4) = 25*24*23*22 \).
  – Alternative way of thinking:
    • There are 25! ways to assign offices to everybody.
    • Out of them, 21! way to assign non-ER offices. We are not interested in what everybody else got – so once the way to assign ER offices is fixed, all sequences with this assignment to ER offices are the same for us.
    • Get \( P(25,4) = \frac{25!}{21!} = 25*24*23*22 \) possible ways to get 4 people offices in ER.

• Winners example:
  – There are 20 students in a programming competition.
  – How many possible ways are there to choose first, second and third winner?
    • 20*19*18

• General formula: \( P(n, r) = \frac{n!}{(n-r)!} \)
Combinations.

• In general, how many ways to pick \( k \) out of \( n \) objects?
  – The number of \( k \)-permutations divided by the number of different permutations of the \( k \) objects themselves.

• **Combinations**: \( C(n,k) \) is number of ways to choose \( k \) objects out of \( n \).
  – “n choose \( k \)”:
    \[
    \binom{n}{k} = \frac{n!}{k!(n-k)!} = C(n,k)
    \]

  – How many ways to choose 4 out of 25 faculty members to get an office in Earth Science?
    • 25! ways to assign offices altogether.
    • 4! ways to assign the first 4 offices (ones in Earth Science).
    • 21! ways to assign offices not in Earth Science.
    • Overall, \( C(25,4) = \frac{25!}{4!21!} \) ways to pick who gets an Earth Science office.

  – How many ways to select a crew of 6 astronauts out of a team of 30 to go to Mars?
    • \( C(30,6) = \binom{30}{6} = \frac{30!}{6!(24!)} = 593,775 \)
ICE CREAM  HAM  RELISH
PANCAKES  KETCHUP  CHEESE
EGGS  CUPCAKES  SOUR CREAM
HOT CHOCOLATE  AVOCADO  SKITTLES

YOU KNOW WHAT'S ACTUALLY REALLY GOOD? [SPACE] FOOD AND [SPACE] FOOD.

HUH. I GUESS I CAN SEE IT.

FUN FACT: IF YOU SAY "YOU KNOW WHAT'S ACTUALLY REALLY GOOD?" IN THE RIGHT TONE OF VOICE, YOU CAN NAME ANY TWO INDIVIDUALLY-GOOD FOODS HERE AND NO ONE WILL CHALLENGE YOU ON IT.
Puzzle: misspelling OSOYOOS

• In the game of Scrabble, players make words out of the pieces they have.
  – Suppose that someone puts the word “OSOYOOS” on the board, using up all her pieces.
  – How many ways could she have had the letters arranged on the rack in front of them?
    • The order of multiple copies of a letter does not matter: switching two S around results in the same sequence, but switching O and S does not.
    • The letters on the rack do not have to form a word.
Puzzle: misspelling OSOYOOS

• Suppose that someone puts the word “OSOYOOS” on the board, using up all her pieces.
• How many ways could she have had the letters arranged on the rack in front of them?
  – There are 7 letters in the word OSOYOOS. If they were all distinct, that would be $7! = 5040$ ways.
  – But there are 4 Os, and 2 Ss, order of which does not matter.
  – There are $4!$ ways to order Os, and $2!$ ways to order Ss.
  – Therefore, the total number of ways to order the letters ignoring the order of Os and Ss is $\frac{7!}{4!2!} = 105$
Puzzle: misspelling OSOYOOS

- Suppose that someone puts the word “OSOYOOS” on the board, using up all her pieces.
- How many ways could she have had the letters arranged on the rack in front of them, such that Ss are not next to each other?
  - First, let’s consider all possible orderings of remaining letters: \( \frac{5!}{4!} \) of them.
  - Now, consider places where S can go: _o_o_y_o_o_ (here, ooyoo are in arbitrary order). There are 6 such places.
  - So there are \( \binom{6}{2} = \frac{6!}{2!4!} \) ways to place Ss.
  - Therefore, the total number of ways to order the letters ignoring the order of Os and Ss and with Ss not next to each other is \( \frac{5!6!}{4!4!2!} = 75 \)
  - Alternatively, consider all orderings with Ss next to each other: there are \( \frac{6!}{4!} = 30 \) of them (treating the “SS” as a single letter).
  - Now, the total is 105-30 = 75.
Combinations with repetition

• Suppose that 10 of you came to the office hour, and we decided to go to Jumping Bean to get some coffee.
• Jumping Bean sells 6 types of coffee drinks: drip coffee, cappucino, espresso, latte, mocca and americano.
• How many different orders can we place, if each gets one coffee drink?
  – That is, how many ways are there to select 11 items, where each item comes from one of the 6 categories?
    • Let’s use one letter for each type:
      • e.g., aacdeellmm stands for 2 americanos, one cappucino, one drip, three espressos, 2 lattes and 2 moccas.
    – Idea: think of a string of letters with “dividers” between different types of drinks:
      • aa|c|d|eee|ll|mm
    – How many ways are there to position the dividers?
      • 6-1= 5 dividers
      • Number of orders + number of dividers: 11+6-1=16 positions.
      • So \( \binom{16}{11} = \binom{16}{5} = 4368 \) possible orders.

• In general, number of ways to select \( r \) objects out of \( n \) categories with repetition is \( \binom{r+n-1}{r} \)
## Summary

<table>
<thead>
<tr>
<th>Selecting $k$ out of $n$ objects</th>
<th>Order matters (permutations)</th>
<th>Order ignored (combinations)</th>
</tr>
</thead>
<tbody>
<tr>
<td>With repetitions</td>
<td>$n^k$</td>
<td>$(k + n - 1 \choose k)$</td>
</tr>
<tr>
<td>Without repetitions</td>
<td>$P(n, k) = \frac{n!}{(n-k)!}$</td>
<td>$(n \choose k)$</td>
</tr>
</tbody>
</table>
Puzzle: playing poker

• There are 52 cards in a standard deck; 4 suites of 13 ranks each.

• In poker, some 5-card combinations (“hands”) are special:
  – For example, a “three of a kind” consists of three cards with the same rank, together with two arbitrary cards.

• How many ways are there to choose (ignoring the order)
  – a three of a kind hand?
  – A two pairs hand?
  – Other hands?...