Admin stuff

• Assignment 3 is posted
  – Due Monday, March 13
Puzzle: all horses are white

• Claim: all horses are white.

• Proof (by induction):
  – P(n): any n horses are white.
  – Base case: P(0) holds vacuously
  – Induction hypothesis: any k horses are white.
  – Induction step: if any k horses are white, then any k+1 horses are white.
    • Take an arbitrary set of k+1 horses. Take a horse out.
      – The remaining k horses are white by induction hypothesis.
    • Now put that horse back in, and take out another horse.
      – Remaining k horses are again white by induction hypothesis.
    • Therefore, all the k+1 horses in that set are white.
  – By induction, all horses are white.
Sums, products and sequences

• How to write long sums, e.g., 1+2+... (n-1)+n concisely?
  – Sum notation (“sum from 1 to n“): \( \sum_{i=1}^{n} i = 1 + 2 + \ldots + n \)
    • If \( n=3 \), \( \sum_{i=1}^{3} i = 1+2+3=6 \).
    • The name “\( i \)” does not matter. Could use another letter not yet in use.

• In general, let \( f: \mathbb{Z} \to \mathbb{R}, m, n \in \mathbb{Z}, m \leq n \).
  – \( \sum_{i=m}^{n} f(i) = f(m) + f(m+1) + \ldots + f(n) \)
    • If \( m=n \), \( \sum_{i=m}^{n} f(i) = f(m) = f(n) \).
    • If \( n=m+1 \), \( \sum_{i=m}^{n} f(i) = f(m) + f(m+1) \)
    • If \( n>m \), \( \sum_{i=m}^{n} f(i) = (\sum_{i=m}^{n-1} f(i)) + f(n) \)
      • Example: \( f(x) = x^2 \). \( 2^2 + 3^2 + 4^2 = \sum_{i=2}^{4} i^2 = 29 \)

• Similarly for product notation (product from m to n):
  – \( \prod_{i=m}^{n} f(i) = f(m) \cdot f(m+1) \cdot \ldots \cdot f(n) = (\prod_{i=m}^{n-1} f(i)) \cdot f(n) \)
  – For \( f(x) = x \), \( 2 \cdot 3 \cdot 4 = \prod_{i=2}^{4} i = 24 \)
  – \( 1 \cdot 2 \cdot \ldots \cdot n = \prod_{i=1}^{n} i = n! \) (n factorial)
Recurrences and sequences

- To define a sequence (of things), describe the process which generates that sequence.
  - **Sequence**: enumeration of objects $s_1, s_2, s_3, \ldots, s_n, \ldots$,
    - Sometimes use notation $\{s_n\}$ for the sequence (i.e., set of elements forming a sequence)
  - **Basis (initial conditions)**: what are the first (few) element(s) in the sequence.
    - $\sum_{i=0}^{0} i = 0$. $\sum_{i=m}^{m} i = m$.
    - $0! = 1$. $1! = 1$.
    - $A_0 = \emptyset$
  - **Recurrence (recursion step, inductive definition)**: a rule to make a next element from already constructed ones.
    - $\sum_{i=m}^{n+1} i = (\sum_{i=m}^{n} i) + (n + 1)$. Here, assume that $m \leq n$
    - $(n+1)! = n! \cdot (n+1)$
    - $A_{n+1} = \mathcal{P}(A_n)$
- Resulting sequences:
  - $m, 2m+1, 3m+3, \ldots$
  - $1, 2, 6, 24, 120, \ldots$
  - $\emptyset, \{\emptyset\}, \{\emptyset, \emptyset\}, \{\emptyset, \{\emptyset\}, \{\emptyset\}, \{\emptyset, \emptyset\}\}, \ldots$
Special sequences

• Arithmetic progression:
  – Sequence: \( c, c + d, c + 2d, c + 3d, \ldots, c + nd, \ldots \)
  – Recursive definition:
    • Basis: \( s_0 = c \), for some \( c \in \mathbb{R} \)
    • Recurrence: \( s_{n+1} = s_n + d \), where \( d \in \mathbb{R} \) is a fixed number.
  – Closed form: \( s_n = c + nd \)
    • Closed forms are very useful for analysis of recursive programs, etc.

• Geometric progression:
  – Sequence: \( c, cr, cr^2, cr^3, \ldots, cr^n, \ldots \)
  – Recursive definition:
    • Basis: \( s_0 = c \), for some \( c \in \mathbb{R} \)
    • Recurrence: \( s_{n+1} = s_n \cdot r \), where \( r \in \mathbb{R} \) is a fixed number.
  – Closed form: \( s_n = c \cdot r^n \)
Fibonacci sequence

• Imagine that a ship leaves a pair of rabbits on an island (with a lot of food).
• After a pair of rabbits reaches 2 months of age, they produce another pair of rabbits, which in turn starts reproducing when reaching 2 months of age...
• How many pairs rabbits will be on the island in n months, assuming no rabbits die?
• Basis: $F_1 = 1$, $F_2 = 1$
• Recurrence: $F_n = F_{n-1} + F_{n-2}$
• Sequence: 1,1,2,3,5,8,13...
• Closed form: $F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$
  – Golden ratio: $\phi$
  – $\phi = \frac{a+b}{a} = \frac{a}{b} = \frac{1+\sqrt{5}}{2}$
Partial sums

• Properties of a sum:
  – \( \sum_{i=m}^{n} f(i) + g(i) = \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i) \)
  – \( \sum_{i=m}^{n} c \cdot f(i) = c \sum_{i=m}^{n} f(i) \)

• Sum of arithmetic progression:
  – \( s_n = c + nd \) for some \( c, d \in \mathbb{R} \)
  – Sequence: \( c, c + d, c + 2d, c + 3d, \ldots, c + nd, \ldots \)
  – Partial sum:
    \[ \sum_{i=0}^{n} s_n = \sum_{i=0}^{n} (c + id) = \sum_{i=0}^{n} c + \sum_{i=0}^{n} id = c(n + 1) + d \sum_{i=0}^{n} i = c(n + 1) + d \frac{n(n+1)}{2} \]

• Sum of geometric progression:
  – \( s_n = c \cdot r^n \) for some \( c, r \in \mathbb{R} \)
  – Sequence: \( c, cr, cr^2, cr^3, \ldots, cr^n, \ldots \)
  – Partial sum:
    • If \( r=1 \), then \( \sum_{i=0}^{n} s_n = c(n + 1) \)
    • If \( r \neq 1 \), then \( \sum_{i=0}^{n} s_n = \frac{cr^{n+1} - c}{r-1} \)
Fractals

• Can use recursive definitions to define fractals
  – And draw them
  – And prove their properties.
• Self-similar: a part looks like the whole.
Fractals in nature

• A fern leaf

• Broccoli

• Mountains

• Stock market

• Heat beat
Mathematical fractals

- Koch curve and snowflake
- Sierpinski triangle, pyramid, carpet
- Hilbert space-filling curve
- Mandelbrot set
Koch curve

- **Basis:** an interval
- **Recursive step:** Replace the inner third of the interval with two of the same length
- ...

![Koch curve stages](image)
Playing with fractals

- Fractal Grower by Joel Castellanos:
Tower of Hanoi game

• Rules of the game:
  – Start with all disks on the first peg.
  – At any step, can move a disk to another peg, as long as it is not placed on top of a smaller disk.
  – Goal: move the whole tower onto the second peg.

• Question: how many steps are needed to move the tower of 8 disks? How about \( n \) disks?