COMP 1002

Logic for Computer Scientists

Lecture 21
Admin stuff

• Assignment 3 is posted
  – Due Monday, March 13

• For the lab this Wednesday, please read definitions in slides for Lecture 19
  – which we did not have time to cover on Tuesday before the midterm
Sum of numbers formula

- **Claim:** for any $n \in \mathbb{N}$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

- **Proof.**
  - Suppose not.
  - Let $S$ be a set of all numbers $n'$ such that $\sum_{i=0}^{n'} i \neq \frac{n'(n'+1)}{2}$. By well-ordering principle, if $S \neq \emptyset$, then there is the least number $k$ in $S$.
  - Case 1: $k=0$. But $\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$. So formula works for $k=0$.
  - Case 2: $k>0$. Then $k-1 \geq 0$.
    - So $\sum_{i=0}^{k} i = (\sum_{i=0}^{k-1} i) + k$.
    - As $k$ is the smallest bad number, the formula works for $k-1$.
    - So $\sum_{i=0}^{k-1} i = \frac{(k-1)k}{2}$
    - Now, $\sum_{i=0}^{k} i = (\sum_{i=0}^{k-1} i) + k = \frac{(k-1)k}{2} + k = \frac{k^2-k+2k}{2} = \frac{k^2+k}{2} = \frac{k(k+1)}{2}$
    - So the formula works for $k>0$, too.
  - Contradiction. So $S$ is empty, thus the formula works for all $n \in \mathbb{N}$.

Gauss' proof:

1 + 2 + ... + 99 + 100 + 100 + 99 + ... + 2 + 1 = 101 + 101 + ... + 101 + 101 = 100*101

So 1+2+ ... + 99 + 100 = $\frac{100*101}{2}$

Works for any $n$, not just $n=100$
Mathematical induction

- Want to prove a statement $\forall x \in \mathbb{N} \ P(x)$.
  - Check that $P(0)$ holds
  - And whenever $P(k)$ does not hold for some $k$, $P(k - 1)$ does not hold either
    - Contradicting well-ordering principle.
    - Contrapositive:
      - if $P(k-1)$ holds for arbitrary $k$,
      - then $P(k)$ also must be true.
  - Conclude that $\forall x \in \mathbb{N} \ P(x)$
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Mathematical Induction principle:
If $P(0) \land \forall k \in \mathbb{N} \ P(k) \rightarrow P(k+1)$ then $\forall x \in \mathbb{N} \ P(x)$
Sum of numbers formula

• Claim: for any $n \in \mathbb{N}$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

• Proof (by induction).
  
  – $P(n)$ is $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ (statement we are proving by induction on $n$)
  
  – Base case: $k=0$. Then $\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$.
  
  – Induction hypothesis: Assume that $\sum_{i=0}^{k-1} i = \frac{(k-1)k}{2}$ for an arbitrary $k > 0$
    
    • That is, for an arbitrary number $k-1 \in \mathbb{N}$
    • Can take $k$ instead of $k-1$, but $k-1$ makes calculations simpler.
  
  – Induction step: show that $P(k-1)$ implies $P(k)$.
    
    • $\sum_{i=0}^{k} i = (\sum_{i=1}^{k-1} i) + k$
    
    • By induction hypothesis, $\sum_{i=1}^{k-1} i = \frac{(k-1)k}{2}$
    
    • Now, $\sum_{i=1}^{k} i = (\sum_{i=1}^{k-1} i) + k = \frac{(k-1)k}{2} + k = \frac{k^2-k+2k}{2} = \frac{k^2+k}{2} = \frac{k(k+1)}{2}$
  
  – By induction, therefore, $P(n)$ holds for all $n \in \mathbb{N}$. 
Changing the base case

• Mathematical Induction principle:
  – \((P(0) \land \forall k \in \mathbb{N} \ P(k) \rightarrow P(k+1)) \rightarrow \forall x \in \mathbb{N} P(x)\)

• What if want to prove it only for \(x \geq a\)?
  – Make \(a\) the base case (when \(a \geq 0\)). For the rest, assume \(k \geq a\).
  – \((P(a) \land \forall k \geq a \ P(k) \rightarrow P(k+1)) \rightarrow \forall x \geq a \ P(x)\)
    • Here, \(\forall x \geq a \ P(x)\) is a shorthand for \(\forall x \in \mathbb{N} \ (x \geq a \rightarrow P(x))\)
    – To prove it works, prove \(P(n')\) where \(n'=n-a\).

• Example: show that for all \(n \geq 4\), \(2^n \geq n^2\)
  – \(P(n)\): \(2^n \geq n^2\)
  – Base case: \(n=4\). \(2^4 = 16 = 4^2\)
  – Induction hypothesis: assume that for an arbitrary \(k \geq a\), \(2^k \geq k^2\)
  – Induction step: show that \(2^k \geq k^2\) implies \(2^{k+1} \geq (k+1)^2\)
    • \(2^{k+1} = 2 \cdot 2^k = 2^k + 2^k \geq k^2 + k^2\)
    • \((k+1)^2 = k^2 + 2k + 1\).
    • Want: \(k^2 + k^2 \geq k^2 + 2k + 1\), so \(k^2 \geq 2k + 1\)
      – Dividing both sides of the inequality by \(k\): \(k \geq 2 + \frac{1}{k}\)
      – Since \(k \geq 4\), and \(2 + \frac{1}{k} \leq 3\), \(2 + \frac{1}{k} \leq 3 < 4 \leq k\). So \(k \geq 2 + \frac{1}{k}\) and thus \(k^2 \geq 2k + 1\)
    • So \(2^{k+1} = 2 \cdot 2^k = 2^k + 2^k \geq k^2 + k^2 \geq k^2 + 2k + 1 = (k+1)^2\)
      – By induction, for all \(n \geq 4\), \(2^n \geq n^2\)

• Corollary: as \(n\) grows, an algorithm running in time \(n^2\) will quickly outperform an algorithm running in time \(2^n\)
Strong induction

• For our coins problem, needed not just $P(k-1)$, but $P(k-3)$, and to look at three cases.

• Mathematical Induction principle:
  – $(P(0) \land \forall k \in \mathbb{N} \ P(k) \rightarrow P(k+1)) \rightarrow \forall x \in \mathbb{N} \ P(x)$

• Strong Induction principle:
  – $\exists b \in \mathbb{N} \ \forall c \in \mathbb{N} \ (0 \leq c \land c \leq b \rightarrow P(c))$
    $\land \forall k > b \ (\forall i \in \{0, \ldots, k - 1\} \ P(i)) \rightarrow P(k)$
    $\rightarrow \forall x \in \mathbb{N} \ P(x)$

• Strong induction seems stronger...
  – But in fact, mathematical induction, strong induction and well-order principles are equivalent to each other.
  – So choose the most convenient one.
Puzzle: coins

• A not-too-far-away country recently got rid of a penny coin, and now everything needs to be rounded to the nearest multiple of 5 cents...
  – Suppose that instead of just dropping the penny, they would introduce a 3 cent coin.
    • Like British three pence.
  – What is the largest amount that cannot be paid by using only existing coins (5, 10, 25) and a 3c coin?

7c
Any number n >7 can be paid with 3,5,10,25 coins (even just 3 and 5).
Strong induction

• **Strong Induction** principle (general form):
  
  - \( (\exists b \in \mathbb{N} \ \forall c \in \mathbb{N} \ (a \leq c \land c \leq b \rightarrow P(c)) \land \forall k > b \ (\forall i \in \{a, \ldots, k-1\} \ P(i)) \rightarrow P(k)) \rightarrow \forall x \in \mathbb{N} \ (x \geq a \rightarrow P(x)) \)

• Coins: \( \forall x \in \mathbb{N}, \text{if } x > 7 \text{ then } \exists y, z \in \mathbb{N} \text{ such that } x = 3y+5z. \)
  
  - \( P(n): \ \exists y, z \in \mathbb{N} \ n = 3y + 5z. \) Also, \( a=8. \)
  
  - Base cases: \( b = 10, \) so \( c \in \{8,9,10\} \)
    - \( n=8. \) \( 8 = 3 \cdot 1 + 5 \cdot 1, \) so \( y=1, \ z=1. \)
    - \( n=9. \) \( 9=3 \cdot 3, \ y=3, \ z=0 \)
    - \( n=10. \) \( 10=5 \cdot 5. \ y=0, \ z=2. \)
  
  - Induction hypothesis: Let \( k \) be an arbitrary integer such that \( k > 10. \) Assume that for all \( i \in \mathbb{N} \) such that \( 8 \leq i < k \) \( \exists y_i, z_i \in \mathbb{N} \ i = 3y_i + 5z_i \)
  
  - Induction step. Show that induction hypothesis implies that \( \exists y, z \in \mathbb{N} \ k = 3y + 5z \)
    - Since \( k \geq b, \ k - 3 \geq a. \) So by induction hypothesis \( \exists y_{k-3}, z_{k-3} \in \mathbb{N} \ k - 3 = 3y_{k-3} + 5z_{k-3}. \) Now take \( z=z_{k-3} \) and \( y=y_{k-3} +1. \) Then \( k = 3y+5z. \)
  
  - By strong induction, get that for all \( x > 7, \exists y, z \in \mathbb{N} \text{ such that } x = 3y+5z. \)
Puzzle: all horses are white

• Claim: all horses are white.

• Proof (by induction):
  – \( P(n) \): any \( n \) horses are white.
  – Base case: \( P(0) \) holds vacuously
  – Induction hypothesis: any \( k \) horses are white.
  – Induction step: if any \( k \) horses are white, then any \( k+1 \) horses are white.
     • Take an arbitrary set of \( k+1 \) horses. Take a horse out.
       – The remaining \( k \) horses are white by induction hypothesis.
     • Now put that horse back in, and take out another horse.
       – Remaining \( k \) horses are again white by induction hypothesis.
     • Therefore, all the \( k+1 \) horses in that set are white.
  – By induction, all horses are white.