



COMP 1002

# Logic for Computer Scientists

Lecture 20



# Admin stuff

- Assignment 3 is posted
  - Due Monday, March 13
- For the lab this Wednesday, please read definitions in slides for Lecture 19
  - which we did not have time to cover on Tuesday before the midterm





## Puzzle: coins



- A not-too-far-away country recently got rid of a penny coin, and now everything needs to be rounded to the nearest multiple of 5 cents...
  - Suppose that instead of just dropping the penny, they would introduce a 3 cent coin.
    - Like British three pence.
  - What is the largest amount that cannot be paid by using only existing coins (5, 10, 25) and a 3c coin?

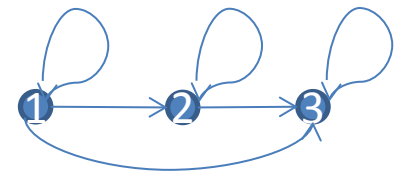
7c

Any number  $n > 7$  can be paid with 3,5,10,25 coins (even just 3 and 5).



# Partial and total orders

- A binary relation  $R \subseteq A \times A$  is an **order** if R is
  - Reflexive  $\forall x \in A R(x, x)$
  - **Anti-symmetric**  $\forall x, y \in A R(x, y) \wedge R(y, x) \rightarrow x = y$
  - Transitive  $\forall x, y, z \in A R(x, y) \wedge R(y, z) \rightarrow R(x, z)$
  - R is a **total order** if  $\forall x, y \in A R(x, y) \vee R(y, x)$ 
    - That is, every two elements of A are related.
    - E.g.  $R_1 = \{(x, y) | x, y \in \mathbb{Z} \wedge x \leq y\}$  is a total order.
    - So is alphabetical order of English words.
    - But not  $R_2 = \{(x, y) | x, y \in \mathbb{Z} \wedge x < y\}$ 
      - not reflexive, so not an order.
  - Otherwise, R is a **partial order**.
    - $SUBSETS = \{(A, B) | A, B \text{ are sets} \wedge A \subseteq B\}$  is a partial order.
      - Reflexive:  $\forall A, A \subseteq A$
      - Anti-symmetric:  $\forall A, B A \subseteq B \wedge B \subseteq A \rightarrow A = B$
      - Transitive:  $\forall A, B, C A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$
      - Not total: if  $A = \{1, 2\}$  and  $B = \{1, 3\}$ , then neither  $A \subseteq B$  nor  $B \subseteq A$
    - $DIVISORS = \{(x, y) | x, y \in \mathbb{N} \wedge x, y \geq 2 \wedge \exists z \in \mathbb{N} y = z \cdot x\}$  is a partial order.
- An order may have **minimal** and **maximal** elements (maybe multiple)
  - $x \in A$  is minimal in R if  $\forall y \in A y \neq x \rightarrow \neg R(y, x)$ 
    - and maximal if  $\forall y \in A y \neq x \rightarrow \neg R(x, y)$
  - $\emptyset$  is minimal in SUBSETS (its unique minimum); universe is maximal (its unique maximum).
  - All primes are minimal in DIVISORS, and there are no maximal elements.





# Well-ordering principle

- Any non-empty subset of natural numbers contains the least element
  - With respect to the usual total order  $x \leq y$
  - Very useful for proofs!
- Coins:  $\forall x \in \mathbb{N}$ , if  $x > 7$  then  $\exists y, z \in \mathbb{N}$  such that  $x = 3y + 5z$ . So any amount  $> 7$  can be paid with 3s and 5s.
  - Suppose, for the sake of contradiction, that there are amounts greater than 7 which cannot be paid with 3s and 5s.
  - *Take a set  $S$  of all such amounts. Since  $S \subseteq \mathbb{N}$ , and we assumed that  $S \neq \emptyset$ , by well-ordering principle  $S$  has the least element. Call it  $n$ .*
  - Now, look at  $n-3$ ; it cannot be paid by 3s and 5s either.
  - Since  $n$  is the least element of  $S$ ,  $n - 3 \leq 7 < n$
  - 3 cases:
    - $n-3 = 7$ . Then  $n=10=2*5$ .
    - $n-3 = 6$ . Then  $n=9=3*3$
    - $n-3 = 5$ . Then  $n=8=3+5$ .
  - In all three cases, got a contradiction.
  - Therefore, for every  $x \in \mathbb{N}$ , if  $x > 7$  then  $x=3y+5z$  for some  $y, z \in \mathbb{N}$ .





# Sums, products and sequences

- How to write long sums, e.g.,  $1+2+\dots+(n-1)+n$  concisely?
  - Sum notation (“sum from 1 to n”):  $\sum_{i=1}^n i = 1 + 2 + \dots + n$ 
    - If  $n=3$ ,  $\sum_{i=1}^3 i = 1+2+3=6$ .
    - The name “ $i$ ” does not matter. Could use another letter not yet in use.
- In general, let  $f: \mathbb{Z} \rightarrow \mathbb{R}$ ,  $m, n \in \mathbb{Z}, m \leq n$ .
  - $\sum_{i=m}^n f(i) = f(m) + f(m+1) + \dots + f(n)$ 
    - If  $m=n$ ,  $\sum_{i=m}^n f(i) = f(m) = f(n)$ .
    - If  $n=m+1$ ,  $\sum_{i=m}^n f(i) = f(m) + f(m+1)$
    - If  $n > m$ ,  $\sum_{i=m}^n f(i) = (\sum_{i=m}^{n-1} f(i)) + f(n)$
    - Example:  $f(x) = x^2$ .  $2^2 + 3^2 + 4^2 = \sum_{i=2}^4 i^2 = 29$
- Similarly for product notation (product from  $m$  to  $n$ ):
  - $\prod_{i=m}^n f(i) = f(m) \cdot f(m+1) \cdot \dots \cdot f(n) = (\prod_{i=m}^{n-1} f(i)) \cdot f(n)$
  - For  $f(x) = x$ ,  $2 \cdot 3 \cdot 4 = \prod_{i=2}^4 i = 24$
  - $1 \cdot 2 \cdot \dots \cdot n = \prod_{i=1}^n i = n!$  ( $n$  factorial)



# Sum of numbers formula

- Claim: for any  $n \in \mathbb{N}$ ,  $\sum_{i=0}^n i = \frac{n(n+1)}{2}$

- Proof.

- Suppose not.

- Let  $S$  be a set of all numbers  $n'$  such that  $\sum_{i=0}^{n'} i \neq \frac{n'(n'+1)}{2}$ . By well-ordering principle, if  $S \neq \emptyset$ , then there is the least number  $k$  in  $S$ .

- Case 1:  $k=0$ . But  $\sum_{i=0}^0 i = 0 = \frac{0(0+1)}{2}$ . So formula works for  $k=0$ .

- Case 2:  $k>0$ . Then  $k - 1 \geq 0$ .

- So  $\sum_{i=0}^k i = (\sum_{i=0}^{k-1} i) + k$ .

- As  $k$  is the smallest bad number, the formula works for  $k-1$ .

- So  $\sum_{i=0}^{k-1} i = \frac{(k-1)k}{2}$

- Now,  $\sum_{i=0}^k i = (\sum_{i=0}^{k-1} i) + k = \frac{(k-1)k}{2} + k = \frac{k^2 - k + 2k}{2} = \frac{k^2 + k}{2} = \frac{k(k+1)}{2}$

- So the formula works for  $k>0$ , too.

- Contradiction. So  $S$  is empty, thus the formula works for all  $n \in \mathbb{N}$ .

Gauss' proof:

$$1 + 2 + \dots + 99 + 100 +$$

$$100 + 99 + \dots + 2 + 1 =$$

$$101 + 101 + \dots + 101 + 101 = 100 \cdot 101$$

$$\text{So } 1+2+ \dots + 99 + 100 = \frac{100 \cdot 101}{2}$$

Works for any  $n$ , not just  $n=100$