COMP 1002

Logic for Computer Scientists

Lecture 19
Admin stuff

• **Midterm March 2\textsuperscript{nd}**.
  – Closed-book.
  – Covers lectures 1 to 16.
    • Mainly labs 1, 2, 3, 4 and assignments 1 and 2.
  – Study guide posted to help you study
    • **not** to bring to the midterm itself.
  – Sample midterm is posted.
    • I will not be posting solutions, but will be happy to give you feedback on yours.

• Assignments 1 and 2 are marked.
  • Except for several assignments 2, will be ready in a couple of hours.
  – Read your feedback on D2L
  – Let me know as soon as possible if you have questions about your mark.

• Lab 5 moved from the week before the break to this Wednesday.
Relations

- A relation is a subset of a Cartesian product of sets.
  - If of two sets (set of pairs), call it a **binary** relation.
  - Of 3 sets (set of triples), **ternary**. Of k sets (set of tuples), **k-ary**

- $A=\{1,2,3\}$, $B=\{a,b\}$
  - $A \times B = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$
  - $R = \{(1,a), (2,b), (3,a), (3,b)\}$ is a relation. So is $R=\{(1,b)\}$.

- $A=\{1,2\}$,
  - $A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$
  - $R=\{(1,1), (2,2)\}$ (all pairs $(x,y)$ where $x=y$)
  - $R=\{(1,1), (1,2), (2,2)\}$ (all pairs $(x,y)$ where $x \leq y$).

- $A=$PEOPLE
  - COUPLES $=\{(x,y) \mid \text{Loves}(x,y)\}$
  - PARENTS $=\{(x,y) \mid \text{Parent}(x,y)\}$

- $A=$PEOPLE, $B=$DOGS, $C=$PLACES
  - WALKS $=\{(x,y,z) \mid x \text{ walks } y \text{ in } z\}$
    - Jane walks Buddy in Bannerman park.
Types of binary relations

• A binary relation $R \subseteq A \times A$ is
  
  – **Reflexive** if $\forall x \in A, R(x, x)$
    
    • Every $x$ is related to itself.
    • E.g. $A=\{1,2\}, R_1 = \{(1,1), (2,2), (1,2)\}$
    • On $A = \mathbb{Z}$, $R_2 = \{(x, y)|x = y\}$ is reflexive
    • But not $R_3 = \{(x, y)|x < y\}$

  – **Symmetric** if $\forall x, y \in A, (x, y) \in R \iff (y, x) \in R$
    
    • $R_1$ and $R_3$ above are not symmetric. $R_2$ is.
    • $A = \mathbb{Z}, R_4 = \{(x, y)|x \equiv y \text{ mod } 3\}$ is symmetric.

  – **Transitive** if $\forall x, y, z \in A, (x, y) \in R \land (y, z) \in R \rightarrow (x, z) \in R$
    
    • $R_1, R_2, R_3, R_4$ are all transitive.
    • $R_5 = \{(x, y)|x, y \in \mathbb{Z} \land x + 1 = y\}$ is not transitive.
    • $\text{PARENT} = \{(x, y)|x, y \in \text{PEOPLE} \land x \text{ is a parent of } y\}$ is not.
    • A **transitive closure** of a relation $R$ is a relation $R^* = \{(x, z)|\exists k \in \mathbb{N} \exists y_0, ..., y_k \in A \ (x = y_0 \land z = y_k \land \forall i \in \{0, ..., k - 1\} R(y_i, y_{i+1})\}$
      
      – That is, can get from $x$ to $z$ following $R$ arrows.
Types of binary relations

• A binary relation \( R \subseteq A \times A \) is
  – **Anti-reflexive** if \( \forall x \in A, \neg R(x, x) \)
    • \( R \) can be neither reflexive nor anti-reflexive.
    • E.g. \( A=\{1,2\}, \ R_6 = \{(1,2)\} \)
      – but not \( R_1 = \{(1,1), (2,2), (1,2)\} \) (reflexive)
      – nor \( R_7 = \{(1,1), (1,2)\} \) (neither)
  • For \( A = \mathbb{Z} \), not \( R_2 = \{(x, y)|x = y\} \)
    – Nor \( R_4 = \{(x, y)|x \equiv y \ mod \ 3\} \)
  • But \( R_3 = \{(x, y)|x < y\} \) is anti-reflexive.
    – So are \( R_5 = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} | x + 1 = y\} \)
    – And \( \text{PARENT} = \{(x, y) \in \text{PEOPLE} \times \text{PEOPLE} | x \ is \ a \ parent \ of \ y\} \)
  – **Anti-symmetric** if \( \forall x, y \in A, (x, y) \in R \land (y, x) \in R \rightarrow x = y \)
    • \( R_1, R_3, R_5, R_6, R_7, \text{PARENT} \) are anti-symmetric. \( R_4 \) is not.
    • \( R_2 \) is both symmetric and anti-symmetric.
    • \( R_8 = \{(1,2), (2,1), (1,3)\} \) is neither symmetric nor anti-symmetric.
Equivalence

• A binary relation \( R \subseteq A \times A \) is an **equivalence** if \( R \) is reflexive, symmetric and transitive.
  
  • E.g. \( A = \{1,2\} \), \( R = \{(1,1), (2,2)\} \) or \( R = A \times A \)
  
  • Not \( R_1 = \{(1,1), (2,2), (1,2)\} \) nor \( R_3 = \{(x, y) | x < y\} \)
  
  • On \( A = \mathbb{Z} \), \( R_2 = \{(x, y) | x = y\} \) is an equivalence
  
  • So is \( R_4 = \{(x, y) | x \equiv y \mod 3 \} \)
    
    – Reflexive: \( \forall x \in \mathbb{Z}, \ x \equiv x \mod 3 \)
    
    – Symmetric: \( \forall x, y \in \mathbb{Z}, \ x \equiv y \mod 3 \rightarrow y \equiv x \mod 3 \)
    
    – Transitive: \( \forall x, y, z \in \mathbb{Z}, \ x \equiv y \mod 3 \land y \equiv z \mod 3 \rightarrow x \equiv z \mod 3 \)

• An equivalence relation partitions \( A \) into **equivalence classes**:
  
  – Intersection of any two equivalence classes is \( \emptyset \)
  
  – Union of all equivalence classes is \( A \).
  
  – \( R_4: \mathbb{Z} = \{x | x \equiv 0 \mod 3\} \cup \{x | x \equiv 1 \mod 3\} \cup \{x | x \equiv 2 \mod 3\} \)
  
  – \( R = A \times A \) gives rise to a single equivalence class.
    \( R = \{(1,1), (2,2)\} \) to two.
Partial and total orders

- A binary relation \( R \subseteq A \times A \) is an order if \( R \) is reflexive, anti-symmetric and transitive.
  - \( R \) is a total order if \( \forall x, y \in A \ R(x, y) \lor R(y, x) \)
    - That is, every two elements of \( A \) are related.
    - E.g. \( R_1 = \{(x, y) | x, y \in \mathbb{Z} \land x \leq y\} \) is a total order.
    - So is alphabetical order of English words.
    - But not \( R_2 = \{(x, y) | x, y \in \mathbb{Z} \land x < y\} \)
      - not reflexive, so not an order.
  - Otherwise, \( R \) is a partial order.

- \( \text{SUBSETS} = \{(A, B) | A, B \text{ are sets } \land A \subseteq B \} \) is a partial order.
  - Reflexive: \( \forall A, A \subseteq A \)
  - Anti-symmetric: \( \forall A, B \ A \subseteq B \land B \subseteq A \rightarrow A = B \)
  - Transitive: \( \forall A, B, C \ A \subseteq B \land B \subseteq C \rightarrow A \subseteq C \)
  - Not total: if \( A = \{1, 2\} \) and \( B = \{1, 3\} \), then neither \( A \subseteq B \) nor \( B \subseteq A \)
- \( \text{DIVISORS} = \{(x, y) | x, y \in \mathbb{N} \land x, y \geq 2 \land \exists z \in \mathbb{N} \ y = z \cdot x\} \) is a partial order.
- \( \text{PARENT} \) is not an order. But \( \text{ANCESTOR} \) would be, if defined so that each person is an ancestor of themselves. It is a partial order.

- An order may have minimal and maximal elements (maybe multiple)
  - \( x \in A \) is minimal in \( R \) if \( \forall y \in A \ y \neq x \rightarrow \lnot R(y, x) \)
    - and maximal if \( \forall y \in A \ y \neq x \rightarrow \lnot R(x, y) \)
  - \( \emptyset \) is minimal in \( \text{SUBSETS} \) (its unique minimum); universe is maximal (its unique maximum).
  - All primes are minimal in \( \text{DIVISORS} \), and there are no maximal elements.
Puzzle: coins

• A not-too-far-away country recently got rid of a penny coin, and now everything needs to be rounded to the nearest multiple of 5 cents...
  – Suppose that instead of just dropping the penny, they would introduce a 3 cent coin.
    • Like British three pence.
  – What is the largest amount that cannot be paid by using only existing coins (5, 10, 25) and a 3c coin?