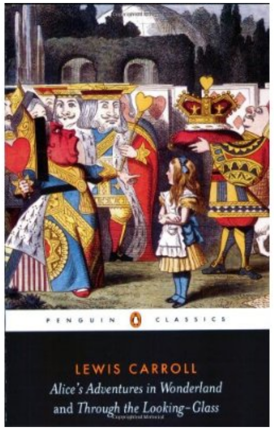
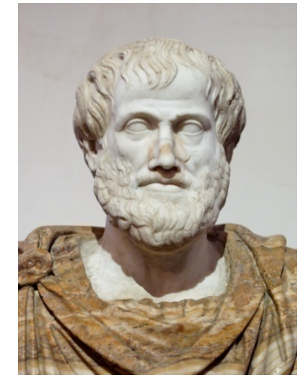


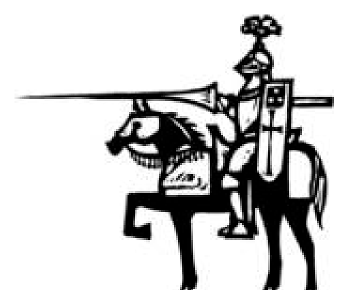



# COMP 1002

# Logic for Computer Scientists

## Lecture 17

B 5 2 J



# Admin stuff

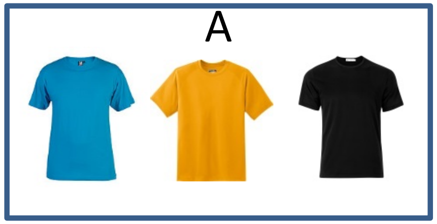
- A2 due Feb 17th.
- Midterm March 2nd.

- Semester break next week!


# Cartesian products

- **Cartesian product** of A and B is a set of all pairs of elements with the first from A, and the second from B:
  - A x B = $\{(x, y) | x \in A,\ y \in B\}$
  - A={1,2,3}, B={a,b}
  - $A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$
  - A={1,2}, $A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$

| | a | b |
|---|---|---|
| 1 | (1,a) | (1,b) |
| 2 | (2,a) | (2,b) |
| 3 | (3,a) | (3,b) |

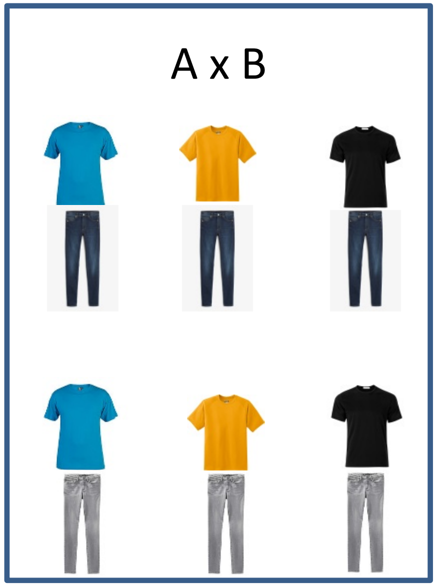




- Order of pairs does not matter, order within pairs does: $A \times B \neq B \times A$ .

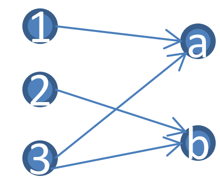
- Number of elements in $A \times B$ is $|A \times B| = |A| \cdot |B|$

- Can define the Cartesian product for any number of sets:
  - $A_1 \times A_2 \times \cdots \times A_k = \{(x_1, x_2, \ldots x_k) |\ x_1 \in A_1 \ldots x_k \in A_k\}$
  - $A = \{1,2,3\}$, $B =${a,b}, C={3,4}
  - $A \times B \times C = \{(1, a, 3), (1, a, 4),\ (1, b, 3), (1, b, 4),$
    $(2, a, 3), (2, a, 4),\ (2, b, 3), (2, b, 4),$
    $(3, a, 3), (3, a, 4),\ (3, b, 3), (3, b, 4)\}$

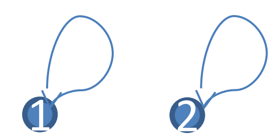



# Relations

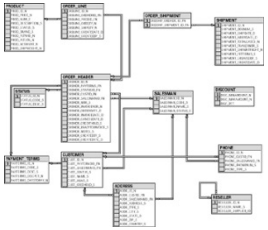
- **A relation** is a subset of a Cartesian product of sets.
  - If of two sets (set of pairs), call it a **binary** relation.
  - Of 3 sets (set of triples), **ternary**. Of k sets (set of tuples), **k-ary**
  - A={1,2,3}, B={a,b}
    - $A \times B = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$
    - R = {(1,a), (2,b),(3,a), (3,b)} is a relation. So is R={(1,b)}.
  - A={1,2},
    - $A \times A = \{(1,1),(1,2),(2,1),(2,2)\}$
    - R={(1,1), (2,2)} (all pairs (x,y) where x=y)
    - R={(1,1),(1,2),(2,2)} (all pairs (x,y) where $x \leq y$).
  - A=PEOPLE
    - COUPLES ={(x,y) | Loves(x,y)}
    - PARENTS ={(x,y) | Parent(x,y)}
  - A=PEOPLE, B=DOGS, C=PLACES
    - WALKS = {(x,y,z) | x walks y in z}
      - Jane walks Buddy in Bannerman park.




Graph of R (bipartite)




Graph of {(1,1),(2,2)}



# Databases and predicates


| Relation R | Predicate P |
|---|---|
| A set of tuples | True/false on a given tuple |
| R=$\{(x_1,\ldots,x_k) \mid P(x_1,\ldots,x_k)$ is true$\}$ | $P(x_1,\ldots,x_k) \equiv (x_1,\ldots,x_k) \in R$ |

- In a database, store relations as tables.
- Then ask queries as predicate logic formulas
  - Return the set of all database elements satisfying the formula.

ProfData

| | A | B | C |
|---|---|---|---|
| 1 | Manrique | Mata-Montero | EN-2033 |
| 2 | Sharene | Bungay | ER-6032 |
| 3 | Antonina | Kolokolova | ER-6033 |

CourseData

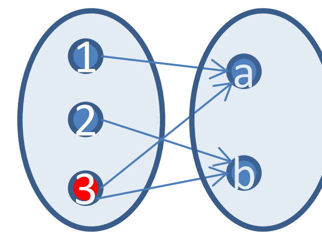
| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | COMP1000 | MWF | 11:00-11:50 | EN-1054 | Mata-Montero |
| 2 | COMP1001 | MWF | 12:00-12:50 | EN-2040 | Bungay |
| 3 | COMP1002 | MTR | 13:00-13:50 | EN-2007 | Kolokolova |

  - “Return first names of all profs who teach MWF “
  - $Q(fn)$:

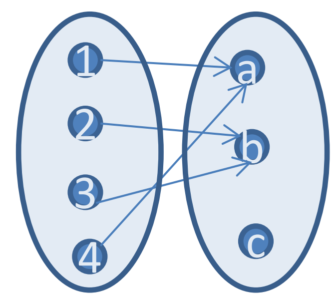
$$\exists\, ln\, \exists\, o\;\; ProfData(fn, ln, o)$$
$$\wedge\, \exists c,t,r\; CourseData(c, \text{"MWF"}, t, r, ln)$$

# Functions

- **A function** $f: X \rightarrow Y$ is a relation on X× $Y$ such that for every $x \in$ X there is at most one $y \in Y$ for which $(x, y)$ is in the relation.
  - Usual notation: $f(x) = y$
    - y is an **image** of x under f.
  - X is the **domain** of f
  - Y is the **codomain** of f
  - **Range** of f (**image** of X under f):
    - $\{y \in Y \mid \exists x \in X, f(x) = y\}$
  - **Preimage** of a given $y \in Y$:
    - $\{x \in X \mid f(x) = y\}$
      - Preimage of b is {2,3}.




This R is not a function

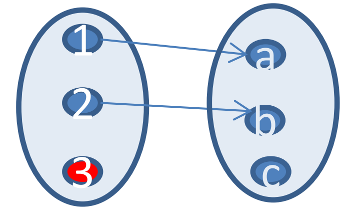



This R is a function with domain {1,2,3,4}, codomain {a,b,c} and range {a,b}

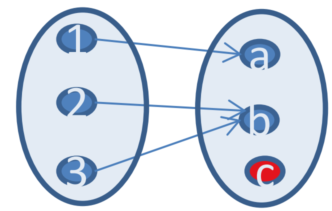

# Functions

- **A function** $f: X \to Y$ is
  - **Total**: $\forall x \in X\ \exists y \in Y\ f(x) = y$
    - f: $\mathbb{Z} \to \mathbb{Z}$
    - $f(x) = x + 1$ is total.
    - $f(x) = \frac{100}{x}$ is not total.
  - **Onto**: $\forall y \in Y\ \exists x \in X\ f(x) = y$
    - $f(x) = x + 1$ is onto over $\mathbb{Z}$, but not over $\mathbb{N}$
    - $f(x) = 5x$ is not onto ($\mathbb{Z}$)
  - **One-to-one:** $\forall x_1, x_2 \in X\ f(x_1) = \mathrm{f}(\mathrm{x}_1) \to x_1 = x_2$
    - $f(x) = x + 1$ is one-to-one.
    - $f(x) = x^2$ is not one-to-one
  - **Bijection**: both one-to-one and onto.
    - $f(x) = x + 1$ is a bijection over $\mathbb{Z}$.

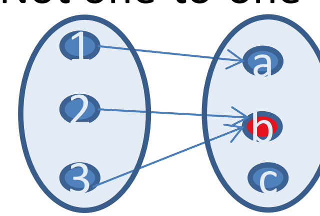
Not total



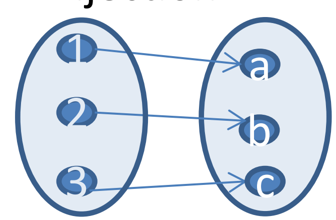

Not onto




Not one-to-one

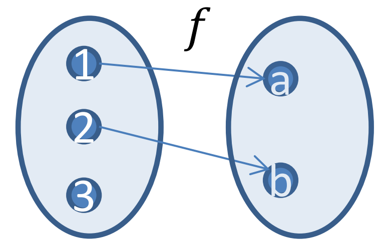


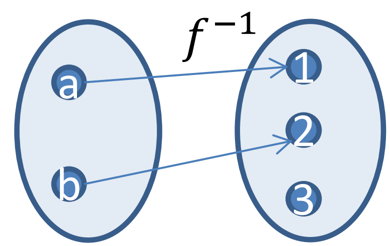

Bijection



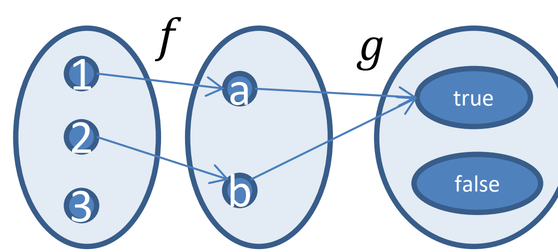
# Functions


- An **inverse** of $f$ is $f^{-1}: Y \to X$, such that $f^{-1}(y) = x$ iff $f(x) = y$
  - $f(x) = x + 1, f^{-1}(y) = y - 1$
  - Only one-to-one functions have an inverse




- **Composition** of $f: X \to Y$ and $g: Y \to Z$ is $g \circ f: X \to Z$ such that $(g \circ f)(x) = g(f(x))$
  - $f(x) = \frac{x}{5},\ g(x) = \lceil x \rceil$, over $\mathbb{R}$
    - $\lceil x \rceil$ is ceiling: x rounded up to nearest integer.
  - $(g \circ f)(x) = g(f(x)) = \left\lceil \frac{x}{5} \right\rceil$
  - $(f \circ g)(x) = f(g(x)) = \frac{\lceil x \rceil}{5}$
  - $(g \circ f)$(12.5)=$\lceil 2.5 \rceil = 3$. $(f \circ g)$(12.5)=13/5= 2.6
    - Order matters!



# Puzzle: the barber club

- In a certain barber's club,
  - Every member has shaved at least one other member
  - No member shaved himself
  - No member has been shaved by more than one member
  - There is a member who has never been shaved.

- *Question: how many barbers are in this club?*