COMP 1002

Intro to Logic for Computer Scientists

Lecture 13
Admin stuff

• Assignments schedule? Split a2 and a3 in two (A2,3,4,5), 5% each. A2 due Feb 17th.

• Midterm date? March 2nd.

• No office hour on Feb 9th
Puzzle 11

• Let

\[ S = \{ x \in \mathbb{N} \mid x \text{ is even} \} \cap \{ x \in \mathbb{N} \mid x \text{ is odd} \} \]

• Prove or disprove:

\[ \forall x \in S, \quad x \text{ does not divide } x^2 \]
Puzzle 11

- Let $S = \{ x \in \mathbb{N} \mid x \text{ is even} \} \cap \{ x \in \mathbb{N} \mid x \text{ is odd} \}$
  - $S = \emptyset$

- Prove or disprove:
  $\forall x \in S, \quad x \text{ does not divide } x^2$
  - Let $P(x) = "x \text{ does not divide } x^2"$
  - To disprove, can give a counterexample
    - Find an element in $S$ such that $P(x)$ is true...
    - But there is no such element in $S$, because there are no elements in $S$ at all!
  - To prove, enough to check that it holds for all elements of $S$.
    - There is none, so it does hold for every element in $S$.
  - Another way: Since $S$ is defined as a subset of natural numbers, can read $\forall x \in S \ P(x)$ as $\forall x \in \mathbb{N} \ (x \in S \rightarrow P(x))$.
    - Since "$x \in S$" is always false, $x \in S \rightarrow P(x)$ is true for every $x \in \mathbb{N}$
  - Call a statement $\forall x \in \emptyset \ P(x)$ **vacuously true.**
Universal Modus Ponens

• All men are mortal
• Socrates is a man
• Therefore, Socrates is mortal

• All cats like fish
• Molly likes fish
• Therefore, Molly is a cat
Universal Modus Ponens

- $\forall x, P(x) \rightarrow Q(x)$
- $P(a)$
- --------------------
- $Q(a)$

- All men are mortal ($\forall x, \text{Man}(x) \rightarrow \text{Mortal}(x)$)
- Socrates is a man ($\text{Man}(\text{Socrates})$)
- Therefore, Socrates is mortal ($\text{Mortal}(\text{Socrates})$)

- All numbers are either odd or even
- 2 is a number
- Therefore, 2 is either odd or even.

- All trees drop leaves
- Pine does not drop leaves
- Therefore, pine is not a tree
Universal Modus Ponens

• All men are mortal
• Socrates is a man
• Therefore, Socrates is mortal

• All cats like fish
• Molly likes fish
• Therefore, Molly is a cat
Instantiation/generalization

• In general, if $\forall x \in S \ F(x)$ is true for some formula $F(x)$, if you take any specific element $a \in S$, then $F(a)$ must be true.
  – This is called the **universal instantiation** rule.
    • $\forall x \in \mathbb{N} \ (x > -1)$
    • $\therefore \ 5 > -1$

• If you prove $F(a)$ without any assumptions about $a$ other than $a \in S$, then $\forall x \in S, F(x)$
  – This is called **universal generalization**.
Instantiation/generalization

• If you can find an element \( a \in S \) such that \( F(a) \), then \( \exists x \in S, F(x) \)
  – This is called existential generalization.

• Alternatively, if \( \exists x \in S \ F(x) \) is true, then you can give that element of \( S \) for which \( F(x) \) is true a name, as long as that name has not been used elsewhere.
  – This is called the existential instantiation rule.

  • \( \exists x \in \mathbb{N} \ (x - 5 = 0) \)
  • \( \therefore k = 0 + 5 \)
Existential instantiation

• If $\exists x \in S \ F(x)$ is true, then you can give that element of $S$ for which $F(x)$ is true a name, as long as that name has not been used elsewhere.

  — “Let $n$ be an even number. Then $n=2k$ for some $k$”.

  • $\forall x \in \mathbb{N} \ Even(x) \rightarrow \exists y \in \mathbb{N} \ (x = 2 \times y)$

  — Important to have a new name!

  • “Let $n$ and $m$ be two even numbers. Then $n=2k$ and $m=2k$” is wrong!

  • $\forall x_1, x_2 \in \mathbb{N} \ Even(x_1) \land Even(x_2) \rightarrow$

    $\exists y_1, y_2 \in \mathbb{N} \ (x_1 = 2 \times y_1) \land (x_2 = 2 \times y_2)$

  • “Let $n$ and $m$ be two even numbers. Then $n=2k$ and $m=2\ell$”
Other inference rules

• Combining universal instantiation with tautologies, get other types of arguments:

\[ \begin{align*}
& \quad p \to q \quad \cdot \quad \forall x \ P(x) \to Q(x) \quad \text{For any } x, \text{ if } x > 3, \text{ then } x > 2 \\
& q \to r \quad \cdot \quad \forall x \ Q(x) \to R(x) \quad \text{For any } x, \text{ if } x > 2, \text{ then } x \neq 1 \\
\hline
\therefore \quad \forall x \ P(x) \to R(x) \quad \text{For any } x, \text{ if } x > 3, \text{ then } x \neq 1
\end{align*} \]

• (This particular rule is called “transitivity”)
Types of proofs

- Direct proof of $\forall x \ F(x)$
  - Show that $F(x)$ holds for arbitrary $x$, then use universal generalization.
    - Often, $F(x)$ is of the form $G(x) \rightarrow H(x)$
    - Example: A sum of two even numbers is even.

- Proof by cases
  - If can write $\forall x \ F(x)$ as $\forall x (G_1(x) \lor G_2(x) \lor \cdots \lor G_k(x)) \rightarrow H(x)$, prove $(G_1(x) \rightarrow H(x)) \land (G_2(x) \rightarrow H(x)) \land \cdots \land (G_k(x) \rightarrow H(x))$.
  - Example: triangle inequality ($|x + y| \leq |x| + |y|$)

- Proof by contraposition
  - To prove $\forall x \ G(x) \rightarrow H(x)$, prove $\forall x \neg H(x) \rightarrow \neg G(x)$
  - Example: If square of an integer is even, then this integer is even.

- Proof by contradiction
  - To prove $\forall x \ F(x)$, prove $\forall x \neg F(x) \rightarrow FALSE$
  - Example: $\sqrt{2}$ is not a rational number.
  - Example: There are infinitely many primes.
Puzzle: better than nothing

- Nothing is better than eternal bliss
- A burger is better than nothing

Therefore, a burger is better than eternal bliss.

Is there anything wrong with this argument?