Unit 2
Reasoning in propositional logic

Computer Science 1002

Introduction to Logic for Computer Scientists

1

Special types of sentences

- A sentence that has a satisfying assignment is **satisfiable.**
  - Some row in the truth table ends with *True.*
  - Example: $B \to A$

A	B	$B \to A$
True	*True*	*True*
True	*False*	*True*
False	*True*	*False*
False	*False*	*True*

- Sentence is a **contradiction (unsatisfiable):**
  - All assignments are falsifying.
  - All rows end with *False.*
  - Example: $A \wedge \neg A$

A	$A \wedge \neg A$
True	*False*
False	*False*

- Sentence is a **tautology:**
  - All assignments are satisfying
  - All rows end with *True.*
  - Example: $B \to A \vee B$
  - We will see later that to check if an argument is correct need to see if a corresponding formula is a tautology

A	B	$A \vee B$	$B \to A \vee B$
True	*True*	*True*	*True*
True	*False*	*True*	*True*
False	*True*	*True*	*True*
False	*False*	*False*	*True*

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Determining formula type

- How long does it take to check if a formula is satisfiable?
  - If somebody gives you a satisfying assignment, then in time roughly the size of the formula.
    - On a m-symbol formula, take time proportional to m
- What if you don't know a satisfying assignment? How hard it is to find it?
  - Using a truth table: in time proportional to $m * 2^n$ on a length m n-variable formula.
  - Is it fast?...

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Complexity of computation

- Would you still consider a problem really solvable if it takes a **very long time?**
  - Say 10^n steps on an n-symbol string?
  - At a billion (10^9) steps per second (~1GHz)?
  - To process a string of length 100...
  - will take $10^{100}/10^9$ seconds, or ~$3\text{x}10^{72}$ centuries.
  - Age of the universe: about $1.38\text{x}10^{10}$ years.
  - Atoms in the observable universe: 10^{78}-10^{82}.

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Complexity of computation

- What strings do we work with in real life?
-
  - Human DNA has about 3.2×10^9 base pairs
  - A secure key in cryptography: 256 bits
  - Number of Walmart transactions per day: 10^9.
  - URLs searched by Google in 2012: $3\text{x}10^{12}$ with $3.5\text{x}10^9$ searches per day now

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Determining formula type

- How long does it take to check if a formula is satisfiable?
  - Using a truth table: about $m * 2^n$ steps
  - Is it efficient?
    - Not really! Formula with 100 variables is already too big!
    - In software verification: millions of variables!
  - Can we do better?

A million-dollar question!

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The million dollar question

- In Russian, called "perebor" (перебор) problem.
  - "perebor" translates as "exhaustive search".
  - **Question: is it always possible to avoid looking through nearly all potential solutions to find an answer?**
    - Such as all truth assignments for a formula with a truth table.
  - **Are there situations when exhaustive search is unavoidable?**

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The million dollar question

- In English, known as "P vs. NP" problem
  - P stands for "polynomial time" (efficiently) computable".
  - NP is "polynomial time checkable"
    - non-deterministic polynomial-time computable
  - **Question: is everything efficiently checkable also efficiently computable?**
    - **Or do we need to go through nearly all potential solutions?**

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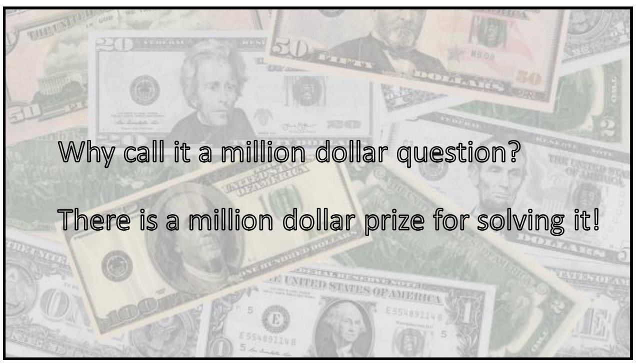

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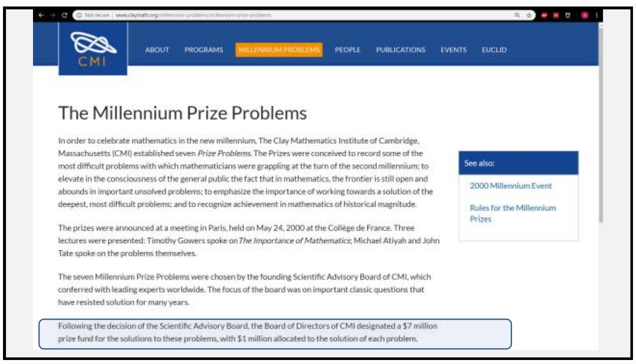

The Millennium Prize Problems

In order to celebrate mathematics in the new millennium, The Clay Mathematics Institute of Cambridge, Massachusetts (CMI) established seven *Prize Problems*. The Prizes were conceived to record some of the most difficult problems with which mathematicians were grappling at the turn of the second millennium; to elevate in the consciousness of the general public the fact that in mathematics, the frontier is still open and abounds in important unsolved problems; to emphasize the importance of working towards a solution of the deepest, most difficult problems; and to recognize achievement in mathematics of historical magnitude.

The prizes were announced at a meeting in Paris, held on May 24, 2000 at the Collège de France. Three lectures were presented: Timothy Gowers spoke on *The Importance of Mathematics*; Michael Atiyah and John Tate spoke on the problems themselves.

The seven Millennium Prize Problems were chosen by the founding Scientific Advisory Board of CMI, which conferred with leading experts worldwide. The focus of the board was on important classic questions that have resisted solution for many years.

Following the decision of the Scientific Advisory Board, the Board of Directors of CMI designated a $7 million prize fund for the solutions to these problems, with $1 million allocated to the solution of each problem.

See also:

2000 Millennium Event

Rules for the Millennium Prizes

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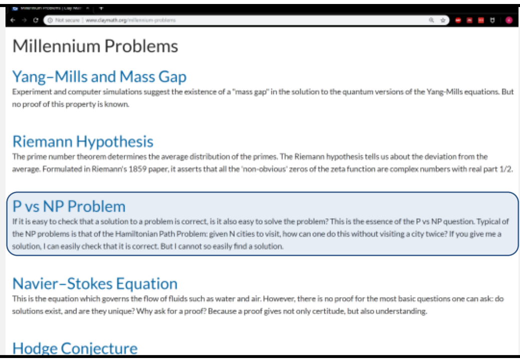

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NP-completeness:

To solve P vs NP, enough to find a way to check if formulas are satisfiable that always works (significantly) faster than truth tables!

Or show that it is impossible.

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- A formula is like a basket of apples.

  formula is a tautology
  =
  All apples in the basket are good.

- Can you check that all apples in a basket are good without looking at every single one?
- Can you do it for every possible basket of apples?

Smell test?

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P vs NP: the million dollar problem.

To show that P=NP, just need to find a smell test for satisfiability.

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Logical equivalence

- Two formulas F and G are **logically equivalent** ($F \Leftrightarrow G$ or $F \equiv G$) if they have the same value for every row in the truth table on their variables.
  - $A \wedge \neg A \equiv FALSE$
    - (same as saying it is a contradiction)
  - $(\neg A \vee B) \equiv (A \rightarrow B)$
  - $(A \leftrightarrow B) \equiv (A \rightarrow B) \wedge (B \rightarrow A)$

A	B	$\neg A \vee B$	$A \rightarrow B$
True	*True*	*True*	*True*
True	*False*	*False*	*False*
False	*True*	*True*	*True*
False	*False*	*True*	*True*

- Useful fact: to prove $F \equiv G$, prove that $F \leftrightarrow G$ is a tautology

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Logical equivalence

Even more useful fact:

- If $F \equiv G$, and F is a subformula of H, then replacing F with G in H results in a formula logically equivalent to H.
  - Recall that a subformula corresponds to a node in a syntax tree.
    - So we can replace a subtree with another for a logically equivalent formula without changing the value of the whole formula.
  - $\mathrm{B} \wedge (A \rightarrow B) \vee C \ \equiv B \wedge (\neg A \vee B) \vee C$,
    - because $(\neg A \vee B) \equiv (A \rightarrow B\ )$
  - $(B \wedge A \rightarrow C) \vee A \equiv \ (\neg(B \wedge A) \vee C) \vee A$
    - Do two replacements: one is renaming variables in $(\neg A \vee B) \equiv (A \rightarrow B\ )$ to some other names (say $(\neg F \vee G) \equiv (F \rightarrow G)$)
    - And the other putting $B \wedge A$ instead of F, and C instead of G.

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Smoking ban puzzle

“I refuse to vote against repealing the ban on smoking in public. “

- Alice says:

Does Alice like smoking or hate it?

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Smoking ban puzzle

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Negating a formula

- Often given a formula A we want to say that it is not true, that is, write a formula equivalent to $\neg A$
  - “It’s sunny **and** cold today”! -- No, it’s not!
- For our brain, hard to understand multiple negations:
  - “I refuse to vote against repealing the ban on smoking in public. “
- How can we simplify such formulas with nested negations?

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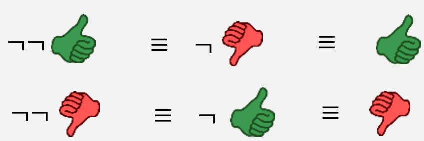
Double negation

- Double negations cancel each other: Law of Excluded Middle.
  - $\neg\neg A \equiv A$
  - “I do not disagree with you” = “I agree with you”



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De Morgan's Laws

- De Morgan's law for AND:

  $\neg\,(\boldsymbol{A} \wedge \boldsymbol{B})$ **is equivalent to** $(\neg \boldsymbol{A} \vee \neg \boldsymbol{B})$

- De Morgan's law for OR:

  $\neg\,(\boldsymbol{A} \vee \boldsymbol{B})$ **is equivalent to** $(\neg \boldsymbol{A} \wedge \neg \boldsymbol{B})$

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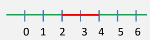
De Morgan's laws: examples

– Let A be "it's sunny" and B "it's cold".
  - "It's sunny **and** cold today"! -- No, it's not! 
  - That could mean 
    - – No, it's not sunny.
    - – No, it's not cold. 
    - – No, it's neither sunny nor cold. 
  - In all of these scenarios, "It's **not** sunny **or not** cold" is true.

– Let A be "x < 2", B be "x > 4".



  - "Either $x < 2$ or $x > 4$" – No, it is not!
  - Then $2 \leq x \leq 4$

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Negating "if .. then"

– Let A be "I play" and B "I win".
  - $A \to B$: "If I play, then I win"
  - Equivalent to $\neg A \vee B$: "Either I do not play, or I win".

– Negation: $\neg(A \to B)$: "It is not so that if I play then I win".
  - By de Morgan's law: $\neg(\neg A \vee B) \equiv (\neg\neg A \wedge \neg B)$
  - By double negation: $(\neg\neg A \wedge \neg B) \equiv (A \wedge \neg B)$
  - So negation of "If I play then I win" is "I play and I don't win".

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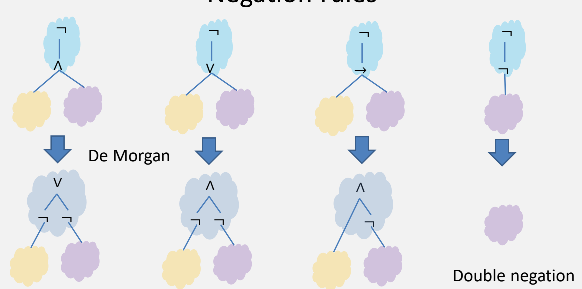
Negation rules



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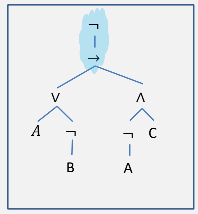
Negating long formulas.

- Start with the syntax tree.
- Starting from the top, keep applying De Morgan's laws and double negation rules.
- Stop when all negations are on variables.

- $\neg\,((A \vee \neg B) \rightarrow\ (\neg A \wedge C))$



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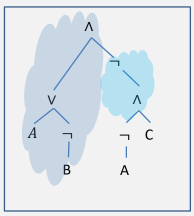
  - $(A \vee \neg B) \wedge \neg(\neg A \wedge C)$ (negating →)



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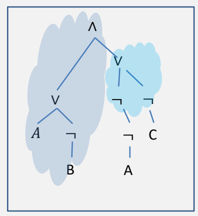

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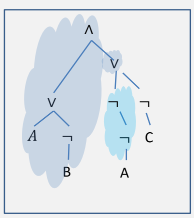

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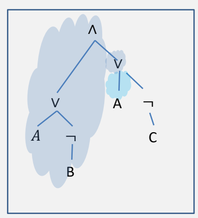
  - $(A \vee \neg B) \wedge (\neg\neg A \vee \neg C)$ (de Morgan)
  - $(A \vee \neg B) \wedge (A \vee \neg C)$ (removing ¬¬)



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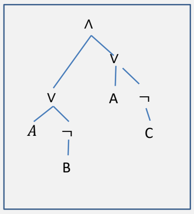

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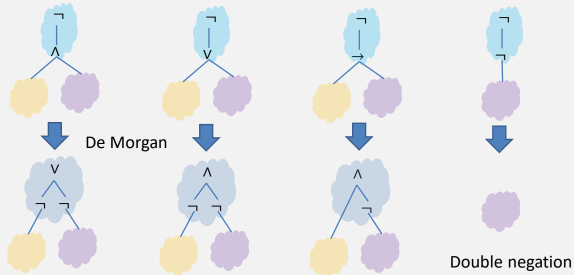
Negation rules



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Knights and knaves again

- On a mystical island, there are two kinds of people: knights and knaves. Knights always tell the truth. Knaves always lie.

- Puzzle: You see three islanders talking to each other, Arnold, Bob and Charlie.
  - You ask Arnold "Are you a knight?", but can't hear what he answered.
  - Bob pitches in: "Arnold said that he is a knave!"
  - and Charlie interjects "Don't believe Bob, he's lying".
  - Out of Bob and Charlie, who is a knight/knave?

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  - Bob pitches in: "Arnold said that he is a knave!" and
  - Charlie interjects "Don't believe Bob, he's lying".
  - Out of Bob and Charlie, who is a knight and who is a knave?

- Look at the sentence "I am a knave". Who of the knights/ knaves can say this?
- If A is "Arnold is a knight" and S is "I am a knave", when is $S \leftrightarrow A$ (what Arnold said is true if and only if he is a knight).
- But also "I am a knave" is the same as saying $\neg A$
- $A \leftrightarrow \neg A$ is a contradiction: it is false no matter what A is.
- So Bob must be lying: Bob is a knave. And Charlie is a knight.

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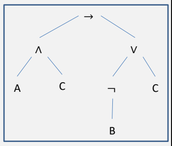
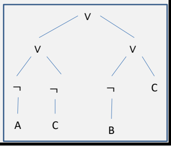
Simplifying formulas

- $A \wedge C \ \rightarrow \neg B \vee C$
  - By $(\mathrm{F} \rightarrow G) \equiv \ (\neg F \vee G)$
    - equivalent to $\neg(A \wedge C) \vee (\neg B \vee C)$
  - De Morgan's law
    - $\neg(A \wedge C)$ is equivalent to $(\neg A \ \vee \neg C)$
- Can we simplify this formula further?





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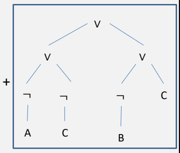
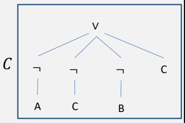
Repeated operations: Associativity rule

- Now get $(\neg A \vee \neg C) \vee (\neg B \vee C)$
  - When you have 3+4+5+6, the result is the same whether you do (3+4)+(5+6) or ((3+4)+5)+6, since all operations are +
  - In fact you'd probably write it without parentheses
  - Similarly, the order in which you do multiplications in 3*4*5*6 does not matter.
- The same applies when we have repeated $\vee$ or repeated $\wedge$ (but not the mix of the two!)
  - Can write $(\neg A \vee \neg C) \vee (\neg B \vee C)$ as $\neg A \vee \neg C \vee \neg B \vee C$
  - We can draw this on the syntax tree as one big $\vee$ or $\wedge$
    - Just as we saw when we talked about Boolean circuits.





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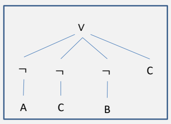
Changing the order: commutativity rule

- We simplified $A \wedge C \ \rightarrow \neg B \vee C$ to $\neg A \vee \neg C \vee \neg B \vee C$
  - Can we go any further?
  - In arithmetic, 3+4+5 = 4+5+3, etc
  - The answer remains the same if you change order
  - As long as terms are part of the same "big" sum or product
- In logic, also can change the order of subformulas in a big $\vee$ or big $\wedge$
  - Let's rewrite $\neg A \vee \neg C \vee \neg B \vee C$ as $\neg A \vee \neg B \vee \neg C \vee C$
  - But $\neg C \vee C$ is always true! So the whole formula is a tautology.



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Logical equivalences

- For any logic formulas A, B, C,
  - like in arithmetic (with $\vee$ as +, $\wedge$ as *)

$A \vee B \equiv B \vee A$	$A \wedge B \equiv B \wedge A$	Commutativity
$(A \vee B) \vee C \equiv A \vee (B \vee C)$	$(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$	Associativity
$(A \vee B) \wedge C \equiv (A \wedge C) \vee (B \wedge C)$		
– And unlike arithmetic	$(A \wedge B) \vee C \equiv \ (A \vee C) \wedge (B \vee C)$	Distributivity (2)

- Properties of $TRUE$ and $FALSE$

$TRUE \vee A \equiv TRUE.$	$TRUE \wedge A \equiv A$
$FALSE \ \vee A \equiv A.$	$FALSE \wedge A \equiv FALSE$
$A \vee \neg A \ \equiv TRUE$	$A \wedge \neg A \equiv \ FALSE$
$A \vee A \equiv A$	$A \wedge A \equiv A$

- Double negation: $\neg\neg A \equiv A$
- De Morgan's laws:
  - $\neg(A \vee B) \equiv \neg A \wedge \neg B$
  - $\neg(A \wedge B) \equiv \neg A \vee \neg B$

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Simplification with distributivity example

- $\neg\left((A \vee \neg B) \rightarrow \ (\neg A \wedge C)\right)$

$\equiv (A \vee \neg B) \wedge \neg(\neg A \wedge C)$ (negating $\rightarrow$)

$\equiv (A \vee \neg B) \wedge (\neg\neg A \vee \neg C)$ (de Morgan's law)

$\equiv (A \vee \neg B) \wedge (A \vee \neg C)$ (double negation)

- Can now simplify further, if we want to.

$\equiv A \vee (\neg B \wedge \neg C)$ (distributivity: factor out A)

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Laws of logic can prove equivalence

- Every Boolean function can be written as a formula.
  - Moreover, as a CNF or DNF formula, using only $\vee, \wedge, \neg$
    - So for every Boolean function we can build a formula to compute it just out of $\vee, \wedge, \neg$
    - Connectives $\rightarrow$ and $\leftrightarrow$ are syntactic sugar.
  - Though there are many formulas with the same truth table, for each truth table there is just one canonical CNF and one canonical DNF.
- So the logic equivalences (laws of logic) can be used to find out whether any two formulas are equivalent.
  - Use laws of logic to convert both formulas to their canonical CNFs (or canonical DNFs).
  - Check if you got the same answer.

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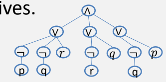
Complete set of connectives

- CNFs only use ¬,∨,∧, yet any formula can be converted into a CNF
  - Any truth table can be coded as a CNF
- Call a set of connectives which can be used to express any truth table **a functionally complete set** of connectives.
  - So ¬,∨,∧, is a complete set of connectives.



- Is this as small as complete sets of connectives can be?

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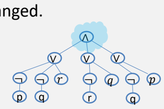
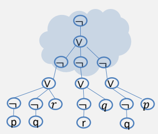
Complete set of connectives

- In fact, ¬,∨ is already complete. So is ¬,∧ .
  - By DeMorgan's, $(A \vee B) \equiv \neg(\neg A \wedge \neg B)$ No need for ∨!
- But ∧,∨ is not complete: cannot do ¬ with just ∧,∨.
  - Because when both inputs have the same value,
  - both ∧,∨ leave them unchanged.
    - $A \wedge A \equiv A \vee A \equiv A$





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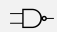
Complete set of connectives

- There is a complete set with just one connective: NAND (NotAND)



  - Also called *Sheffer stroke*, written | and defined by this truth table:

A	B	A | B
True	True	False
True	False	True
False	True	True
False	False	True

- To show that NAND is enough, show how to simulate a complete set of connectives (say ¬,∨) using only |
  1. $\neg A \equiv A \mid A$
  2. $A \vee B \equiv \neg(\neg A \wedge \neg B) \equiv (\neg A \mid \neg B) \equiv ((A \mid A) \mid (B \mid B))$
  - In practice, most often stick to ∧,∨,¬ (called De Morgan basis)

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Ways to say $A \to B$ in English

- Lots of ways to say $A \to B$.
  - "If A, then B" ("If A, B")
  - "A implies B"
  - "A only if B"
  - "B unless ¬A"
  - "B if A" ("B when(ever) A")
  - "B follows from A"
  - "A is sufficient for B"
  - "B is necessary for A"
- A: "it's wet", B: "it's slippery". $A \to B$
  - If it's wet, it's slippery
  - It's wet implies that it's slippery
  - It's wet only if it's slippery.
  - It's slippery unless it is not wet.
  - It's slippery when it's wet.
  - Being slippery follows from being wet
  - Being wet is sufficient (to make it) slippery
  - It must be slippery when it is wet

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More on if… then…

- You see the following cards. Each has a letter on one side and a number on the other.

B	5	2	J

- Which cards do you need to turn to check that **if** a card **has a J** on it **then** it **has a 5** on the other side?

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Contrapositive

- Let $A \to B$ be an **implication** (if A then B, A implies B).
  - If a card has a J on one side, then it has 5 on the other.
- Its **contrapositive** is $\neg B \to \neg A$.
  - If a card does not have 5 on one side, then it cannot have J on the other.
- Contrapositive is equivalent to the original implication:

$$A \to B \equiv \neg B \to \neg A.$$

  - This is why we need to check cards with numbers other than 5!

Proof: $\neg B \to \neg A \equiv \neg\neg B \vee \neg A \equiv B \vee \neg A \equiv \neg A \vee B$

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Converse and inverse

- Let $A \to B$ be an **implication** (if A then B).
  - if a card has a J on one side then it has a 5 on the other
- Its **converse** is $B \to A$
  - If a card has 5 on one side, then it has J on the other.

J 2

5 B

- Converse is **not equivalent** to the original implication!
  - For $A = true, B = false$, $A \to B$ is false, $B \to A$ is true.
- Converse is **not equivalent** to the negation of $A \to B$
  - $\neg(A \to B) \equiv A \wedge \neg B$.
  - For A=true, B=true, $B \to A$ is true, but $\neg(A \to B)$ is false.
- Converse is equivalent to the **inverse** $\neg A \to \neg B$ of $A \to B$
  - If a card does not have J on one side, it cannot have 5 on the other.

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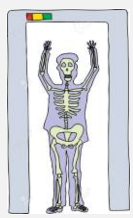
Contrapositive vs. Converse

- "If a person is carrying a weapon, then the airport metal detector will ring".
  - Same as "If the airport metal detector does not ring, then the person is not carrying a weapon".
  - Not the same as: "If the airport metal detector rings, then the person is carrying a weapon."
- "If the person is sick, then the test is positive".
- "If he is a murderer, his fingerprints are on the knife".



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$A \to B$	$\not\equiv$	$B \to A$ Converse of $A \to B$
$\equiv$		$\equiv$
$\neg B \to \neg A$ Contrapositive of $A \to B$	$\not\equiv$	$\neg A \to \neg B$ Inverse of $A \to B$

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If and only if

- $A \leftrightarrow B$ ("A if and only if B", biconditional, also written as "A iff B") is true exactly when both the implication $A \to B$ and its converse $B \to A$ (equivalently, inverse $\neg A \to \neg B$) are true
  - Come to the lab on Monday **if and only if** you are in section 2.
    - If you are in section 2, then come to lab on Monday
    - If you came to lab on Monday, you better be in section 2
      - Equivalently, if you are not in section 2, do not come to Monday lab: come to Wednesday lab instead.
  - Arnold is a knight if and only if he what he said is true.

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Treasure hunt

- In the back of an old cupboard you discover a note signed by a pirate famous for his bizarre sense of humour and love of logical puzzles. In the note he wrote that he had hidden a treasure somewhere on the property. He listed 5 true statements and challenged the reader to use them to figure out the location of the treasure

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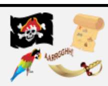
Treasure hunt



1. If this house is next to a lake, then the treasure is not in the kitchen
2. If the tree in the front yard is an elm, then the treasure is in the kitchen
3. This house is next to a lake
4. The tree in the front yard is an elm, or the treasure is buried under the flagpole
5. If the tree in the back yard is an oak, then the treasure is in the garage.

Where is the gold?

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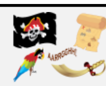
Treasure hunt

- In the back of an old cupboard you discover a note signed by a pirate famous for his bizarre sense of humour and love of logical puzzles. In the note he wrote that he had hidden a treasure somewhere on the property. He listed 5 true statements and challenged the reader to use them to figure out the location of the treasure

Where is the gold?

65

Treasure hunt



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5. If the tree in the back yard is an oak, then the treasure is in the garage.

66

Treasure hunt

1. If this house is next to a lake, then a treasure is not in the kitchen
2. If the tree in the front yard is an elm, then the treasure is in the kitchen
3. This house is next to a lake
4. The tree in the front yard is an elm, or the treasure is buried under the flagpole
5. If the tree in the back yard is an oak, then the treasure is in the garage.

- A: this house is next to a lake.
- B: the treasure is in the kitchen
- C: The tree in front is elm
- D: the treasure is under the flagpole.
- E: The tree in the back is oak
- F: The treasure is in the garage

Too many variables for a nice truth table...

1. If A then not B
2. If C then B
3. A
4. C or D
5. If E then F

1. $A \to \neg B$
2. $C \to B$
3. A
4. C ∨ D
5. E → F

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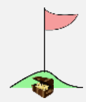
Natural deduction

- A: this house is next to a lake.
- B: the treasure is in the kitchen
- C: The tree in front is elm
- D: the treasure is under the flagpole.
- E: The tree in the back is oak
- F: The treasure is in the garage

1. $A \to \neg B$
2. $C \to B$
3. A
4. C ∨ D
5. E → F
6. $\neg B$
7. $\neg C$
8. D

- If house is next to the lake then the treasure is not in the kitchen
- The house is next to the lake
- Therefore, the treasure is not in the kitchen.



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Arguments and validity

- An **argument**, in logic, is a sequence of propositional statements.
  - Called **argument form** when statements are formulas involving variables.
- The last statement in the sequence is called the **conclusion**. All the rest are **premises**.
- An argument is **valid** if whenever all premises are true, the conclusion is also true.
  - So if premises are $P_1, \dots, P_n$, and conclusion is C,

The argument is valid ⟺ (If and only if) $P_1 \wedge P_2 \wedge \cdots P_n \to C$ is a tautology

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Treasure hunt

Premises

1. If this house is next to a lake, then a treasure is not in the kitchen
2. If the tree in the front yard is an elm, then the treasure is in the kitchen
3. This house is next to a lake
4. The tree in the front yard is an elm, or the treasure is buried under the flagpole
5. If the tree in the back yard is an oak, then the treasure is in the garage.

Conclusion

6. The treasure is under the flagpole.

Argument

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Arguments and validity

- Arguments are often written in this format:
  - Symbol ∴ is pronounced "therefore"

P_1
P_2
⋮
P_n
___
$\therefore C$

- If house is next to the lake then the treasure is not in the kitchen
- The house is next to the lake

___
∴ the treasure is not in the kitchen

If $x > 3$, then $x > 2$
If $x > 2$, then $x \neq 1$
$x > 3$
___
$\therefore\ x \neq 1$

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Arguments and validity

- Valid argument: "AND of premises → conclusion" is a tautology

– If house is next to the lake then the treasure is not in the kitchen
– The house is next to the lake
____
∴ the treasure is not in the kitchen

Valid argument:
$((p \to q) \wedge p \to q$
is a tautology

If $x > 3$, then $x > 2$
If $x > 2$, then $x \neq 1$
$x > 3$
____
$\therefore\ x \neq 1$

Valid argument:
$(p \to q) \wedge (q \to r) \wedge p \to r$
is a tautology

If $x > 3$, then $x > 2$
If $x > 2$, then $x \neq 1$
$x \neq 1$
____
$\therefore\ x > 3$

Invalid argument!
$(p \to q) \wedge (q \to r) \wedge r \to p$
is a not a tautology!
False when r is true, and p and q are both false.

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Natural deduction

- A: this house is next to a lake.
- B: the treasure is in the kitchen
- C: The tree in front is elm
- D: the treasure is under the flagpole.
- E: The tree in the back is oak
- F: The treasure is in the garage

1. $A \to \neg B$
2. $C \to B$
3. A
4. C ∨ D
5. E → F
6. $\neg B$
7. $\neg C$
8. D

- If house is next to the lake then the treasure is not in the kitchen
- The house is next to the lake
- Therefore, the treasure is not in the kitchen.

How do we get the intermediate steps?

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Rules of inference

- Just like we used equivalences to simplify a formula instead of writing truth tables
- Can apply tautologies of the form $\mathrm{F} \to G$
  – so that if F is an AND of several formulas derived so far, then we get G, and add G to premises.
  – Such as $((p \to q) \wedge p) \to q$
- Keep going until we get the conclusion.

- If house is next to the lake then the treasure is not in the kitchen
- The house is next to the lake
- Therefore, the treasure is not in the kitchen.

- Here, p is "the house is next to the lake", and q is "the treasure is not in the kitchen".

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Modus ponens

- The main rule of inference, given by the tautology $(p \to q) \wedge p \to q$, is called **Modus Ponens**.
  - "Method of affirming" in Latin
  - If house is next to the lake then the treasure is not in the kitchen
  - The house is next to the lake

  ____
  ∴ the treasure is not in the kitchen

  - If Socrates is a man, then Socrates is mortal
  - Socrates is a man

  ____
  ∴ Socrates is mortal

  - If $x > 2$, then $x \neq 1$
  - $x > 2$

  ____
  $\therefore\ x \neq 1$

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Modus ponens and friends

- There are several rules related to modus ponens
  - Technically not modus ponens, but easily equivalent
    – Since $p \to q \equiv \neg p \vee q \equiv \neg q \to \neg p$
  - Most textbooks consider them separate rules; we don't.
    – But if somebody asks you specifically "what is modus ponens", that's the first rule below.

- $p \to q$
- p

____
∴ q

Modus Ponens

- $\neg q \to \neg p$
- p

____
∴ q

Modus Tollens

- $\neg p \vee q$
- p

____
∴ q

Disjunctive syllogism

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Resolution rule and friends

- Another group of equivalent rules is the resolution rule (that we will see a lot in the next lecture)
  - And its friend hypothetical syllogism

Resolution

- $\neg p \vee q$
- $\neg q \vee r$

$\therefore \neg p \vee r$

Hypothetical syllogism

- $p \to q$
- $q \to r$

$\therefore p \to r$

- If $x > 3$ then $x > 2$
- If $x > 2$ then $x > 1$

$\therefore$ If $x > 3$ then $x > 1$

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Auxiliary rules

- These are short "common-sense" rules. You don't need to know them by name, just be able to use them.

- p
- q

$\therefore p \wedge q$

If derived both p and q, can conclude $p \wedge q$

- $p \wedge q$

$\therefore p$

If $p \wedge q$ is true, then in particular p is true

- p

$\therefore p \vee q$

If p is true, then $p \vee q$ is true for any possible q

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Proof vs. disproof

- To disprove that something is always true, enough to give just one scenario where it is false (find a falsifying assignment).
  - To disprove that $A \to B \equiv B \to A$
    - Take $A = true, B = false$,
    - Then $A \to B$ is false, but $B \to A$ is true.
  - To disprove that $B \to A \equiv \neg(A \to B)$
    - Take A=true, B=true
    - Then $B \to A$ is true, but $\neg(A \to B)$ is false.
  - I have classes every day! – No, you don't have classes on Saturday
  - Women don't do Computer Science! – Me?

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False premises

- An argument can still be valid when some of its premises are false.
  - Remember, contradiction implies anything.
- Bertrand Russell: "If 2+2=5, then I am the pope"

Puzzle: can you see how to prove this?

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False premises

- An argument can still be valid even when some of the premises are false.
  - Remember, "contradiction implies A" is true for any A.
- Bertrand Russell: "If 2+2=5, then I am the pope"

Puzzle: can you see how to prove this?

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False premises

- Bertrand Russell: "If 2+2=5, then I am the pope"
- Proof:
  - Suppose 2+2=5
  - If 2+2=5, then 1=2 (subtract 3 from both sides).
  - So 1=2 (by modus ponens)
  - Me and the pope are two people.
  - Since 1=2, me and the pope are one person.
  - Therefore, I am the pope!

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Valid and invalid arguments

- An argument is **valid** if whenever all premises $P_1, \ldots, P_n$ are true, the conclusion C is also true.
  - That is, $P_1 \wedge P_2 \wedge \cdots P_n \rightarrow C$ is a tautology
    - $P_1 \wedge P_2 \wedge \cdots P_n \rightarrow C$ should be true no matter what values propositional variables in premises and conclusion take.

  - False premise does not make an argument invalid.
  - Alternatively, there can be an invalid argument with true premises and/or conclusion.
  - The only impossible combination is a valid argument with true premises and false conclusion.

if $2+2=4$ *then* $3+3=6$
$3+3=6$
____
$\therefore 2+2=4$

Invalid argument
$(p \rightarrow q) \wedge q \rightarrow p$
is not a tautology!

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Inconsistent statements

- A list of logic statements that cannot all be true at the same time is called **inconsistent.**
  - That is, an AND of these statements is a contradiction (unsatisfiable).
  - $p, \neg p$ are inconsistent, because $p \wedge \neg p$ is a contradiction
  - $(p \vee q),\ (p \vee \neg q),\ \neg p$ are inconsistent, since $(p \vee q) \wedge (p \vee \neg q) \wedge \neg p$ is unsatisfiable.
  - "I have a lab only on Tuesday. I don't have a class on Tuesday. I have a class and a lab on the same day."
  - "If the sky is clear then the sun is shining. If it is night, then the sky is clear. It is night and the sun is not shining".

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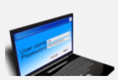
Inconsistent statements

- From a policy rules :
  - You need to change your password every 6 month you otherwise you will get locked out of your account.
  - If your device tries to access the system unsuccessfully several times, you will be locked out of your account.



- From common sense:
  - While I am changing the password on my computer, my phone would be trying to read mail and so will be unsuccessfully accessing my account!

- So the policy rules, common sense and not being locked out of my account are inconsistent! ☹

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Inconsistent premises

- What if the premises are inconsistent, that is, they contradict each other?
  - Then anything can be a conclusion!
  - In particular, a contradiction (such as the constant FALSE) can be a conclusion of a valid argument if and only if its premises are inconsistent

Today is Sunday
Today is not Sunday
____
$\therefore$ 2+2=5

$p \vee q$
$p \vee \neg q$
$\neg p$
______
$\therefore FALSE$

bears are black or bears are white
bears are black or bears are not white
bears are not black
______
$\therefore$ *it is cold outside*

  - Any argument with inconsistent premises is valid (and useless).

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Proof vs. disproof

- To prove that something is (always) true:
  - Make sure it holds in every scenario
    - $\neg B \rightarrow \neg A \equiv A \rightarrow B$, because $\neg B \rightarrow \neg A \equiv \neg\neg B \vee \neg A \equiv B \vee \neg A \equiv \neg A \vee B \equiv A \rightarrow B$
    - So $(\neg B \rightarrow \neg A) \leftrightarrow (A \rightarrow B)$ is a tautology.
    - I have classes every day that starts with T. I have classes on Tuesday and Thursday (and Monday, but that's irrelevant).
  - Or assume it does not hold, and then get something strange as a consequence:
    - To show A is true, enough to show $\neg A \rightarrow FALSE$.
    - Suppose there are finitely many prime numbers. What divides the number that's a product of all primes +1?

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Natural deduction vs. Truth tables

- It was faster to solve the puzzle using natural deduction than writing a truth table.
- But is it always better?
- The answer is...

Nobody knows!

- It is a very closely related to the question of how fast can one check if something is a tautology.

And that's a million dollar question!

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Automated provers

- How to make an automated prover which checks whether a formula is a tautology?
  - And so can check if an argument is valid, etc.

- Truth tables:
  - easy to program, but proofs are huge.
- Natural deduction:
  - proofs might be smaller than a truth table
    - Are they always? Good question...
  - even if there is a small proof, how can we find one quickly?
    - Nobody knows...

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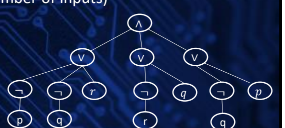
Automated provers

$C \vee x$
$D \vee \neg x$
$\therefore C \vee D$

- Work with the CNF formulas
  - $\wedge$ on top (can have single input, no inputs, many inputs)
  - All children of that $\wedge$ are $\vee$ (also any number of inputs)
    - Called clauses
  - Each child of each $\vee$ is a literal
    - a variable, or
    - single $\neg$ and then a variable.



- Use only the **resolution rule**
  - Basis for many practical provers (SAT solvers)
  - Used in scheduling, verification, etc...

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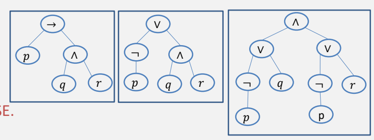
Converting any formula to CNF

- Rewrite it with only $\wedge, \vee, \neg$ using equivalences
  $A \rightarrow B \equiv \neg A \vee B$ as well as $A \leftrightarrow B \equiv (\neg A \vee B) \wedge (A \vee \neg B)$
- Propagate negations to variables, getting rid of double negations.
- Convert each $F \vee (G \wedge H)$ to $(F \vee G) \wedge (F \vee H)$.
- Example: p $\rightarrow q \wedge r$
  $\equiv \neg p \vee (q \wedge r)$
  $\equiv (\neg p \vee q) \wedge (\neg p \vee r)$



- Special cases:
  - An empty clause () is FALSE.
  - An empty CNF is TRUE.

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CNFs in resolution proofs vs. canonical CNFs

- For proofs, get a formula (much smaller than the truth table)
- and want to get some information about its truth table (is it all false?) without writing the truth table out.
  - CNFs on which we do resolution are tiny in comparison to their truth tables, and we hope to find out if they are contradictions faster than writing their truth tables.
- For canonical CNFs, get a truth table, and want a formula computing it.
  - We have the truth table (and so can figure out if f is 0 everywhere) before even starting to write a canonical CNF!
  - If f is 0 everywhere, its canonical CNF has as many clauses as lines in the truth table.

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Resolution rule revisited

- Suppose the premises are clauses $(C \vee x)$, $(D \vee \neg x)$
  - Where C, D are both ∨ of literals, possibly empty
- Now, use the following special form of the resolution rule:

$C \vee x$	$y \vee \neg z \vee w$	$y \vee w \vee \neg z$	w
$D \vee \neg x$	$u \vee \neg w$	$\neg z \vee \neg w$	$\neg w$
$\therefore C \vee D$	$\therefore y \vee \neg z \vee u$	$\therefore y \vee \neg z$	$\therefore FALSE$

- Ignore the order of literals in a clause and remove duplicates.

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Finding inconsistencies: resolution refutation

- Remember that a list of logic statements $A_1, \dots, A_n$ is *inconsistent* if and only if
  - $A_1 \wedge A_2 \wedge \dots \wedge A_n$ is unsatisfiable (contradiction)
  - From $A_1, \dots, A_n$ as premises can derive $FALSE$ as conclusion.

$p \vee q$
$p \vee \neg q$
$\neg p$
$\therefore FALSE$

- When $A_1, \dots, A_n$ are all CNFs,
  - can check that $A_1, \dots, A_n$ are inconsistent by repeatedly applying the resolution rule to (the clauses in) $A_1, \dots, A_n$ until FALSE (empty clause) is derived.

w
$\neg w$
$\therefore FALSE$

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Resolution refutation

$C \vee x$
$D \vee \neg x$
$\therefore C \vee D$

- **Resolution refutation**: a proof that a CNF $A_1 \wedge \dots \wedge A_n$ is a contradiction by applying resolution rule repeatedly to get FALSE
  - If FALSE (empty clause) is derived
    - then $A_1 \wedge \dots \wedge A_n$ is unsatisfiable (contradiction).
    - Any scenario makes at least one clause (and so $A_1 \wedge \dots \wedge A_n$) false
  - If reached the point when there are no more clauses to derive by the resolution rule
    - Then $A_1 \wedge \dots \wedge A_n$ is not a contradiction (satisfiable).

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Resolution refutation

$C \vee x$
$D \vee \neg x$
$\therefore C \vee D$

- Start with $A_1 \wedge \dots \wedge A_n$ (viewed as a list of clauses)
- At every step
  - pick two clauses (original or derived) sharing a variable
    - In one clause negated and in another not negated, such as $(C \vee x), (D \vee \neg x)$
  - Derive a new clause $(C \vee D)$ and add it to the list
    - Nothing ever goes away! No cancellations!
- Repeat until FALSE is derived
  - or cannot derive anything new

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Resolution refutation

$C \vee x$
$D \vee \neg x$
$\therefore C \vee D$

- Start with $A_1 \wedge \cdots \wedge A_n$ (viewed as a list of clauses)
- At every step
  - pick two clauses (original or derived) sharing a variable
    - In one clause negated and in another not negated, such as $(C \vee x), (D \vee \neg x)$
  - Derive a new clause $(C \vee D)$ and add it to the list
    - Nothing ever goes away! No cancellations!
- Repeat until FALSE is derived
  - or cannot derive anything new

$(y \vee \neg z) \wedge (\neg y) \wedge (y \vee z)$

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Resolution refutation

$C \vee x$
$D \vee \neg x$
$\therefore C \vee D$

- Start with $A_1 \wedge \cdots \wedge A_n$ (viewed as a list of clauses)
- At every step
  - pick two clauses (original or derived) sharing a variable
    - In one clause negated and in another not negated, such as $(C \vee x), (D \vee \neg x)$
  - Derive a new clause $(C \vee D)$ and add it to the list
    - Nothing ever goes away! No cancellations!
- Repeat until FALSE is derived
  - or cannot derive anything new

$(y \vee \neg z) \wedge (\neg y) \wedge (y \vee z)$

$(\neg z)$ (z)

$FALSE$

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Resolution refutation

$C \vee x$
$D \vee \neg x$
$\therefore C \vee D$

- Start with $A_1 \wedge \cdots \wedge A_n$ (viewed as a list of clauses)
- At every step
  - pick two clauses (original or derived) sharing a variable
    - In one clause negated and in another not negated, such as $(C \vee x), (D \vee \neg x)$
  - Derive a new clause $(C \vee D)$ and add it to the list
  - Repeat until FALSE is derived
  - or cannot derive anything new

1. $(y \vee \neg z)$
2. $(\neg y)$
3. $(y \vee z)$
4. $(\neg z)$ from 1. and 2. (from $(y \vee \neg z)$ and $(\neg y)$)
5. (z) from 2. and 3. (from $(y \vee z)$ and $(\neg y)$)
6. FALSE from 4. and 5. (from $(\neg z)$ and (z))

Note that $(\neg y)$ was used twice.

$(y \vee \neg z) \wedge (\neg y) \wedge (y \vee z)$

$(\neg z)$ (z)

$FALSE$

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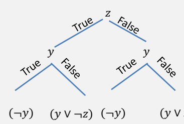
Decision trees

$C \vee x$
$D \vee \neg x$
$\therefore C \vee D$

- Refutation (deriving FALSE):
  - shows that all clauses cannot be made true at the same time.
- An "upside-down" view: decision tree.
  - For every truth assignment, some clause is false.



$(y \vee \neg z) \wedge (\neg y) \wedge (y \vee z)$

$(\neg z)$ (z)

$FALSE$

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$(x \vee \neg y \vee z) \wedge (\neg x \vee \neg z) \wedge (y \vee \neg z) \wedge (\neg y \vee \neg z) \wedge (x \vee y) \wedge (\neg x \vee z)$

$C \vee x$
$D \vee \neg x$
$\therefore C \vee D$

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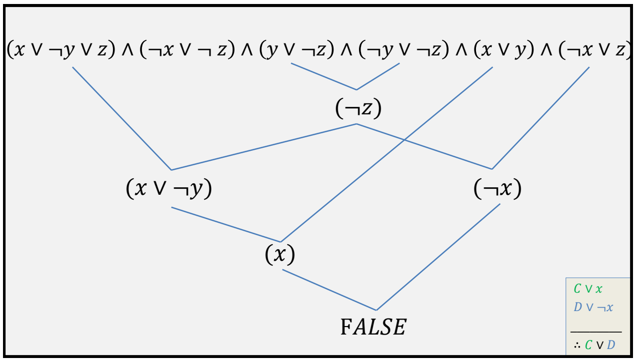

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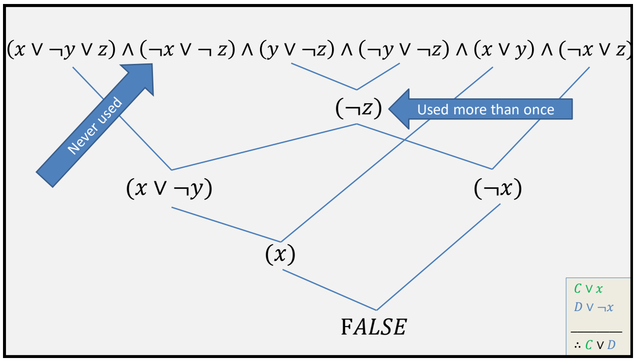

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Pens puzzle

- Suppose that nobody in our class carries more than 10 pens.
- There are 148 students in our class.

- Prove that there are at least 2 students in our class who carry the same number of pens.
  - In fact, there are at least 14 who do

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Pens puzzle

- Suppose that nobody in our class carries more than 10 pens.
- There are 148 students in our class.

- Prove that there are at least 2 students in our class who carry the same number of pens.
  - In fact, there are at least 14 who do

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Pigeonhole Principle

- Suppose that nobody in our class carries more than 10 pens. There are 148 students in our class.
- Prove that there are at least 2 students in our class who carry the same number of pens.
  - In fact, there are at least 14 who do.

- The Pigeonhole Principle:
  - If there are n pigeons
  - And n-1 pigeonholes
  - Then if every pigeon is in a pigeonhole
  - At least two pigeons sit in the same hole

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Pigeonhole Principle

- Suppose that nobody in our class carries more than 10 pens. There are 148 students in our class. Prove that there are at least 2 students in our class who carry the same number of pens.

The Pigeonhole Principle:
If there are n pigeons and n-1 pigeonholes
Then if every pigeon is in a pigeonhole
At least two pigeons sit in the same hole

- Applying to our problem:
  - n-1 = 11 possible numbers of pens
    - (from 0 to 10)
  - Even with n=12 people, there would be 2 who have the same number.
  - If there were less than 14, say 13 for each scenario, total would be 143.
  - Note that it does not tell us which number or who these people are!

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Pigeonhole Principle

- The Pigeonhole Principle:
  - If there are n pigeons
  - And n-1 pigeonholes
  - Then if every pigeon is in a pigeonhole
  - At least two pigeons sit in the same hole

- It is possible that some holes are empty, and other have more than two pigeons.
  - But at least one hole should have more than one pigeon.
- Also works for any $m < n$ holes, with n pigeons

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Resolution and Pigeons

- It is not that hard to write the Pigeonhole Principle as a tautology
- But we can prove that resolution has trouble with this kind of reasoning
  - the smallest resolution proof of this tautology is on the order of 2^n!
- By contrast, natural deduction (and you!) can figure it out fairly quickly
  - though it is not straightforward.
- So some things resolution cannot prove fast.
  - What can it prove quickly? Big open problem!

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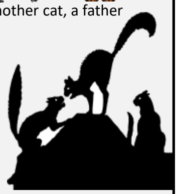
Meow-stery

- One evening there was a cat fight in a family consisting of a mother cat, a father cat, and their son and daughter kittens.
- One of these four cats attacked and bit another!
- One of the cats watched the fight.
- The other one hissed at the fighters.



- These are the things we know for sure:
  - 1. The watcher and the hisser were not of the same gender.
  - 2. The oldest cat and the watcher were not of the same gender.
  - 3. The youngest cat and the victim were not of the same gender.
  - 4. The hissing cat was older than the victim.
  - 5. The father was the oldest of the four.
  - 6. The attacker was not the youngest of the four.
- Which nasty cat was the attacker?

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Natural deduction vs. resolution

- Natural deduction and resolution both can be used to show that a formula is a tautology
  - For resolution, show that the negation of a formula is a contradiction
  - Both can also be used to derive conclusions from premises.
- We now know that resolution proofs can be essentially as big as truth tables: Pigeon Hole Principle.
  - For natural deduction, nobody knows.

- Somehow, in practice, it is easier to get resolution to work than natural deduction.
  - Why??? Great question. Not for the lack of trying.

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Resolution refutations for verifying that an argument is valid

- An argument with premises $P_1, P_2, \dots, P_n$ and conclusion C is valid if and only if $P_1 \wedge P_2 \wedge \cdots \wedge P_n \to C$ is a tautology
  - It is invalid if there is a scenario where $P_1, P_2, \dots, P_n$ are all true, but C is false.
- To verify that an argument is valid
  - That $P_1 \wedge P_2 \wedge \cdots \wedge P_n \to C$ is a tautology,
- Check that $\neg(P_1 \wedge P_2 \wedge \cdots \wedge P_n \to C)$ is a contradiction.
  - That no scenario makes both $P_1 \wedge P_2 \wedge \cdots \wedge P_n$ and $\neg C$ true.

$\neg(P_1 \wedge P_2 \wedge \cdots \wedge P_n \to C) \equiv P_1 \wedge P_2 \wedge \cdots \wedge P_n \wedge \neg C$

1. Convert all premises and $\neg C$ to CNFs
2. Then do resolution on $P_1 \wedge P_2 \wedge \cdots \wedge P_n \wedge \neg C$
3. If could derive $FALSE$, then the argument was valid.

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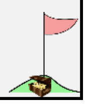
Treasure hunt

1. If this house is next to a lake, then a treasure is not in the kitchen
2. If the tree in the front yard is an elm, then the treasure is in the kitchen
3. This house is next to a lake
4. The tree in the front yard is an elm, or the treasure is buried under the flagpole
5. If the tree in the back yard is an oak, then the treasure is in the garage.

- A: this house is next to a lake.
- B: the treasure is in the kitchen
- C: The tree in front is elm
- D: the treasure is under the flagpole.
- E: The tree in the back is oak
- F: The treasure is in the garage

$A \to \neg B$
$C \to B$
A
C ∨ D
E → F

$\therefore\ D$



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Verifying validity

1. $A \to \neg B$
2. $C \to B$
3. A
4. C ∨ D
5. E → F
6. $\neg B$ (from 1. and 3. by modus ponens)
7. $\neg C$ (from contrapositive of 2. and 6. by modus ponens)
8. D (from 7. and 4 converted to $\neg C \to D$ by modus ponens)

- A: this house is next to a lake.
- B: the treasure is in the kitchen
- C: The tree in front is elm
- D: the treasure is under the flagpole.
- E: The tree in the back is oak
- F: The treasure is in the garage

Let's check that this argument is valid. That is, that $(A \to \neg B) \wedge (C \to B) \wedge A \wedge (C \vee D) \wedge (E \to F) \to D$ is a tautology

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Verifying validity

1. If this house is next to a lake, then a treasure is not in the kitchen
2. If the tree in the front yard is an elm, then the treasure is in the kitchen
3. This house is next to a lake
4. The tree in the front yard is an elm, or the treasure is under the flagpole
5. If the tree in the back yard is an oak, then the treasure is in the garage.

- A: this house is next to a lake.
- B: the treasure is in the kitchen
- C: The tree in front is elm
- D: the treasure is under the flagpole.
- E: The tree in the back is oak
- F: The treasure is in the garage

Argument

$A \to \neg B$
$C \to B$
A
C ∨ D
E → F

$\therefore\ D$

Tautology iff argument is valid

$(A \to \neg B) \wedge (C \to B) \wedge (A) \wedge (C \vee D) \wedge (E \to F) \to D$

Contradiction iff argument is valid

$(A \to \neg B) \wedge (C \to B) \wedge (A) \wedge (C \vee D) \wedge (E \to F) \wedge \neg D$

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Verifying validity

1. If this house is next to a lake, then a treasure is not in the kitchen
2. If the tree in the front yard is an elm, then the treasure is in the kitchen
3. This house is next to a lake
4. The tree in the front yard is an elm, or the treasure is under the flagpole
5. If the tree in the back yard is an oak, then the treasure is in the garage.

- A: this house is next to a lake.
- B: the treasure is in the kitchen
- C: The tree in front is elm
- D: the treasure is under the flagpole.
- E: The tree in the back is oak
- F: The treasure is in the garage

Argument

$A \to \neg B$
$C \to B$
A
C ∨ D
E → F

$\therefore\ D$

Premises and negated conclusion in CNF

1. $\neg A \vee \neg B$
2. $\neg C \vee B$
3. A
4. C ∨ D
5. $\neg E \vee F$
6. $\neg D$

Contradiction iff argument is valid

$(A \to \neg B) \wedge (C \to B) \wedge (A) \wedge (C \vee D) \wedge (E \to F) \wedge \neg D$

Contradiction iff argument is valid (in CNF)

$(\neg A \vee \neg B) \wedge (\neg C \vee B) \wedge (A) \wedge (C \vee D) \wedge (\neg E \vee F) \wedge \neg D$

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Verifying validity

1. If this house is next to a lake, then a treasure is not in the kitchen
2. If the tree in the front yard is an elm, then the treasure is in the kitchen
3. This house is next to a lake
4. The tree in the front yard is an elm, or the treasure is under the flagpole
5. If the tree in the back yard is an oak, then the treasure is in the garage.

- A: this house is next to a lake.
- B: the treasure is in the kitchen
- C: The tree in front is elm
- D: the treasure is under the flagpole.
- E: The tree in the back is oak
- F: The treasure is in the garage

$$(\neg A \lor \neg B) \land (\neg C \lor B) \land (A) \land (C \lor D) \land (\neg E \lor F) \land \neg D$$

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Verifying validity

1. If this house is next to a lake, then a treasure is not in the kitchen
2. If the tree in the front yard is an elm, then the treasure is in the kitchen
3. This house is next to a lake
4. The tree in the front yard is an elm, or the treasure is under the flagpole
5. If the tree in the back yard is an oak, then the treasure is in the garage.

- A: this house is next to a lake.
- B: the treasure is in the kitchen
- C: The tree in front is elm
- D: the treasure is under the flagpole.
- E: The tree in the back is oak
- F: The treasure is in the garage

$$(\neg A \lor \neg B) \land (\neg C \lor B) \land (A) \land (C \lor D) \land (\neg E \lor F) \land \neg D$$

$(\neg A \lor \neg C)$

(C)

$(\neg C)$

$FALSE$

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Deriving conclusions using resolution

- How do we actually derive a conclusion using resolution?
  - Like finding the treasure.
- Suppose the conclusion is just one clause or a literal
  - otherwise derive each clause of the CNF equivalent to the conclusion separately
- Take premises converted to CNF, and keep applying resolution rule until get a conclusion clause
  - Such as a clause containing a single variable stating "the treasure is here"

$$(\neg A \lor \neg B) \land (\neg C \lor B) \land (A) \land (C \lor D) \land (\neg E \lor F)$$

$(\neg A \lor \neg C)$

$(\neg C)$

(D)

Note that in this case there is no way to get FALSE: The premises by themselves are not inconsistent!

We can derive some more clauses: $(\neg B), (B \lor D)$, $(\neg A \lor D)$, and then we have to stop.

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Stereotypes puzzle

- Susan is 28 years old, single, outspoken, and very bright. She majored in philosophy. As a student she was deeply concerned with issues of discrimination and social justice and also participated in anti-pipeline demonstrations.

  Please rank the following possibilities by how likely they are. List them from least likely to most likely. Susan is:

1. a kindergarden teacher
2. works in a bookstore and takes yoga classes
3. an active feminist
4. a psychiatric social worker
5. a member of an outdoors club
6. a bank teller
7. an insurance salesperson
8. a bank teller and an active feminist

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