1. Sets, relations and functions

(a) Disprove by giving a counterexample that for any sets $A, B, C$, $A \cap (B \cup C) = (A \cap B) \cup C$.

(b) Prove that $(B - A) \cup (C - A) = (B \cup C) - A$. First show that every element on the left side belongs to the right side, then show that every element from the right side belongs to the left side, using definitions of union and difference.

(c) Let $g : A \to B$ and $f : B \to C$ be one-to-one functions, where $A, B, C$ are arbitrary sets. Prove that their composition $f \circ g : A \to C$ is a one-to-one function.

2. Boolean algebras

For this question you need the axioms of Boolean algebra:

\[
\begin{align*}
  x + y &= y + x & \text{Commutativity} \\
  (x + y) + z &= x + (y + z) & \text{Associativity} \\
  x \cdot (y + z) &= x \cdot y + x \cdot z & \text{Distributivity} \\
  x + 0 &= x & \text{Identity} \\
  x + \bar{x} &= 1 & \text{Inverse} \\
  x \cdot 1 &= x & \\
  x \cdot \bar{x} &= 0 & \\
  0 \neq 1 &
\end{align*}
\]

(a) Write a formula $\bar{x} + y \cdot \bar{z}$ in both the language of propositional logic (using variables $p, q, r$ for $x, y, z$) and in the language of set theory (using variables $A, B, C$ for $x, y, z$).

(b) Let $x, y$ be elements of a Boolean algebra. Prove using only axioms above that $(x \cdot \bar{y}) + x = x$. (For simplicity, also assume that you already proven that for any element $z$ of Boolean algebra, $z + z = z$).

3. Cardinalities of sets

Consider Tarski worlds with two kinds of pieces (squares and triangles) of three colours (red, green, blue), for the total of 6 distinct kinds of pieces. In this exercise, we will look at Tarski worlds with boards of arbitrary size.

(a) How many possible $1 \times 1$ Tarski world boards are there with these 6 pieces? How many $2 \times 2$ such boards would that be?

(b) How many possible $n \times n$ Tarski boards are there? Hint: to make it easier, think of a $n \times n$ board as a string of length $n^2$, listing cells row after row.

(c) Show that the total number of possible Tarski world boards, for all sizes, is countable. Hint: you can first prove that a larger set is countable, using the hint from the previous subquestion.

(d) Now, let’s look at sets of boards. Some such sets can be described by formulas: for example, we can talk of a set of all boards (of arbitrary size) which have a red square on them. Here, you will show that only very few sets of boards can be described by formulas like this, by showing that the set of such sets of boards is uncountable. That is, prove by diagonalization that the set of sets of Tarski boards is an uncountable set.