COMP 1002

Intro to Logic for Computer Scientists

Lecture 9
Puzzle 8

• Suppose that nobody in our class carries more than 10 pens.
• There are 70 students in our class.
• Prove that there are at least 2 students in our class who carry the same number of pens.
  – In fact, there are at least 7 who do.
Pigeonhole Principle

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  - And \( n-1 \) pigeonholes
  - Then if every pigeon is in a pigeonhole
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• Applying to our problem:
  – \( n-1 = 11 \) possible numbers of pens (from 0 to 10)
  – Even with \( n=12 \) people, there would be 2 who have the same number.
  – If there were less than 7, say 6 for each scenario, total would be 66.
  – Note that it does not tell us which number or who these people are!
Resolution and Pigeons

• It is not that hard to write the Pigeonhole Principle as a tautology
• But we can prove that resolution has trouble with this kind of reasoning
  – the smallest resolution proof of this tautology is exponential size!
• By contrast, natural deduction (and you!) can figure it out fairly quickly
  – though it is not straightforward.
• The problem is that resolution cannot count.
  – But ability to count makes things harder...
Meow-stery

- One evening there was a cat fight in a family consisting of a mother cat, a father cat, and their son and daughter kittens.
- One of these four cats attacked and bit another!
- One of the cats watched the fight.
- The other one hissed at the fighters.

- These are the things we know for sure:
  - 1. The watcher and the hisser were not of the same sex.
  - 2. The oldest cat and the watcher were not of the same sex.
  - 3. The youngest cat and the victim were not of the same sex.
  - 4. The hissing cat was older than the victim.
  - 5. The father was the oldest of the four.
  - 6. The attacker was not the youngest of the four.

- Which nasty cat was the attacker?
Boolean functions and circuits

• What is the relation between propositional logic we studied and logic circuits?
  – View a formula as computing a function (called a Boolean function),
    • inputs are values of variables,
    • output is either true (1) or false (0).
  – For example, $\text{Majority}(x, y, z) = \text{true}$ when at least two out of $x, y, z$ are true, and false otherwise.
  – Such a function is fully described by a truth table of its formula (or its circuit: circuits have truth tables too).
Boolean functions and circuits

• What is the relation between propositional logic and logic circuits?
  – So both formulas and circuits “compute” Boolean functions – that is, truth tables.
  – In a circuit, can “reuse” a piece in several places, so a circuit can be smaller than a formula.
    • Still, most circuits are big!
  – $\text{Majority}(x, y, z) \text{ is } (x \land y) \lor (x \land z) \lor (y \land z)$
Canonical CNF

• Every truth table (Boolean function) can be written as a CNF:
  – Take every falsifying assignment
    • Say,  $A = \text{False}$, $B = \text{True}$, $C=\text{False}$.
  – Write it as a formula which is true only on this assignment:
    • $\neg A \land B \land \neg C$
  – To say that this assignment does not happen, write its negation:
    • $\neg (\neg A \land B \land \neg C) \equiv (A \lor \neg B \lor C)$
  – Take an AND of these for all falsifying assignments
    • It is equivalent to the original formula.
    • And it is a CNF! Called the **canonical CNF** of this formula.
Canonical DNF

• So for every formula, there is a unique canonical CNF (and a truth table, and a Boolean function).
• And for every possible truth table (a Boolean function), there is a formula (the canonical CNF).

• A negation of a CNF is an OR of ANDs of literals. It is called a **DNF (disjunctive normal form)**.
  – To make a canonical DNF from a truth table:
  – take all satisfying assignments.
  – Write each as an AND of literals, as before.
  – Then take an OR of these ANDs.
Complete set of connectives

- CNFs only have $\neg, \lor, \land$, yet any formula can be converted into a CNF
  - Any truth table can be coded as a CNF
- Call a set of connectives which can be used to express any formula a **complete set** of connectives.
  - In fact, $\neg, \lor$ is already complete. So is $\neg, \land$.
    - By DeMorgan, $(A \lor B) \equiv \neg(\neg A \land \neg B)$ No need for $\lor$!
  - But $\land, \lor$ is not: cannot do $\neg$ with just $\land, \lor$.
    - Because when both inputs have the same value, both $\land, \lor$ leave them unchanged.
Complete set of connectives

- How many connectives is enough?
  - Just one: NAND (NotAND), also called the Sheffer stroke, written as |
  - \( \neg A \equiv A \land A \)
  - \( A \lor B \equiv \neg(\neg A \land \neg B) \)
    \[ \equiv (\neg A \lor \neg B) \]
    \[ \equiv ( (A \lor A) \lor (B \lor B)) \]
  - In practice, most often stick to \( \land, \lor, \neg \)
Puzzle 9

• Susan is 28 years old, single, outspoken, and very bright. She majored in philosophy. As a student she was deeply concerned with issues of discrimination and social justice and also participated in anti-nuke demonstrations.

*Please rank the following possibilities by how likely they are. List them from least likely to most likely. Susan is:*

1. a kindergarden teacher
2. works in a bookstore and takes yoga classes
3. an active feminist
4. a psychiatric social worker
5. a member of an outdoors club
6. a bank teller
7. an insurance salesperson
8. a bank teller and an active feminist