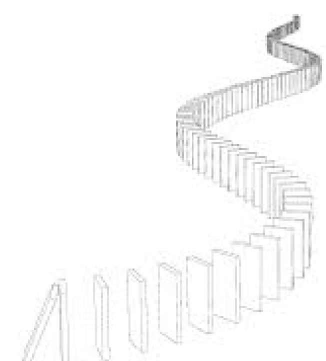
COMP 1002

Logic for Computer Scientists

Lecture 30

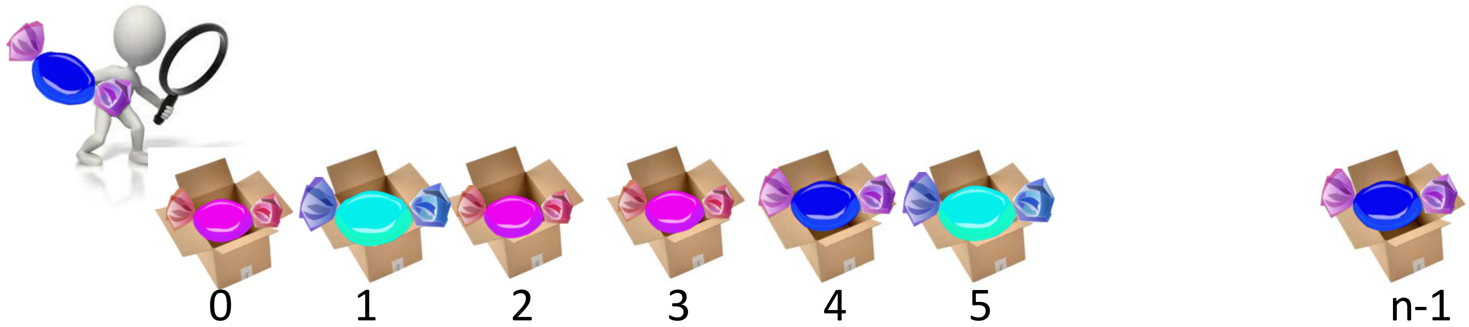
B 5 2 J



Analysis of algorithms

- Putting it all together:
  - Using **logic** to describe what an algorithm is doing
  - and **induction** to show that it does that correctly
  - Using **recurrence** relations to see how long it takes in the worst case.
    - With **O-notation** to talk about the time.
  - and **probabilities/expectation** to try to see how long it might take on average.

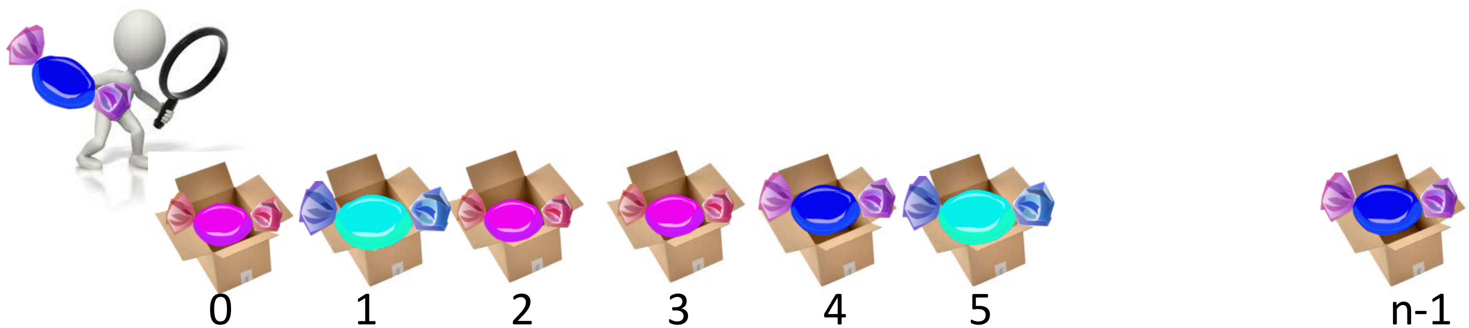
Example: search in an array



- Given:
  - an array A containing n elements, 
  - and a specific item x 

- Goal: find the index of x in A, if x is in A.
  - Which box contains ? Box 4. 

Example: search in an array



- Given:
  - an array A containing n elements, 
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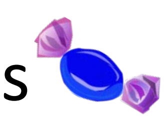
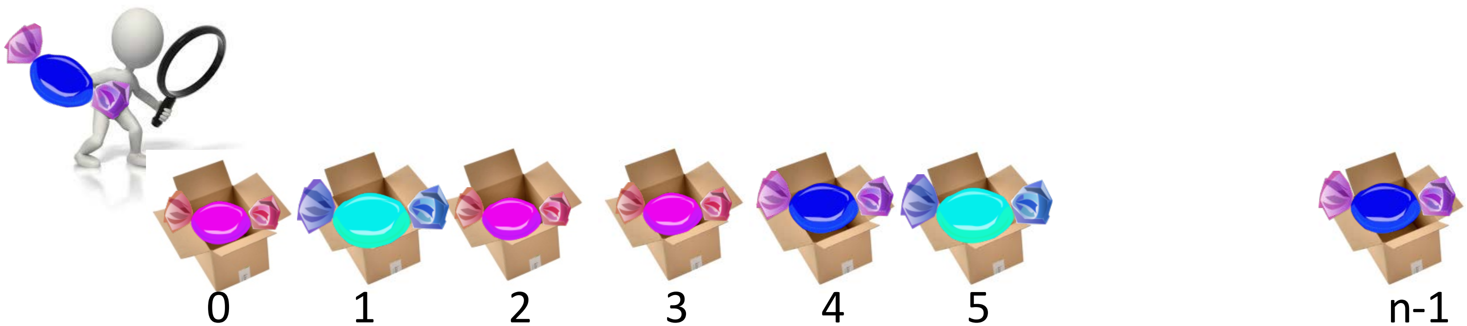
- Goal: find the index of x in A, if x is in A.
  - Which box contains ? Box 4. 

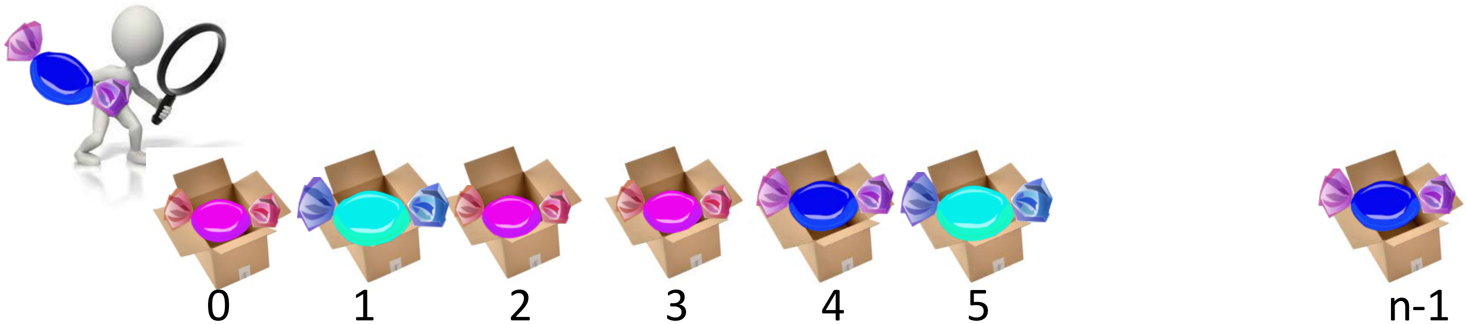
Example: search in an array



- *Precondition*: what should be true before a piece of code (or the whole algorithm) starts
  - E.g.: A is an array of numbers and A is not empty and x is a number.

- *Postcondition*: what should be true after a program (piece of code) finished.
  - E.g. If the program returned value k, then A[k]=x
    - or k=-1, if x is not in A.

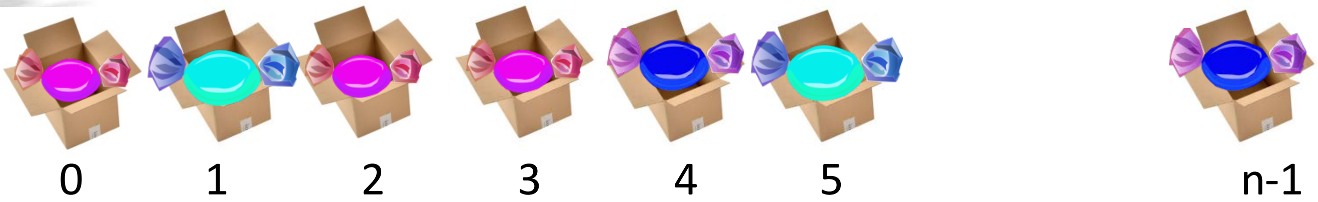
Example: search in an array



- *Precondition*: A is an array containing x

- *Postcondition*: Returned k such that A[k]=x

Example: search in an array



- *Precondition*: A is an array containing x

```
Algorithm arraySearch(A, x)
Input array A of n integers, number x
Output k such that A[k]=x

i = 0
out = -1

while  out < 0 do
        if A[i]  =  x  then
                out  = i
         i = i+1

return out
```

- *Postcondition*: Returned k such that A[k]=x

arraySearch algorithm

Algorithm ***arraySearch*(*A*, *x*)**
Input array ***A*** of ***n*** integers, number x
Output k such that A[k]=x

$\exists i \in \{0 \dots n-1\}\ A[i] = x$

i = 0
out = -1

$\exists i \in \{0 \dots n-1\}\ A[i] = x\ \wedge i = 0 \wedge out = -1$

while ***out*** < ***0*** **do**
    if $A[i] = x$ **then**
        out = ***i***
    i = ***i*+*1***

$A[out] = x$

return ***out***

Program returned k such that A[k]=x

- *A = [5,10,8,7]*
- *x = 8*
- *out = 2*

Loop invariant

- **Loop invariant:** a condition that is true on each iteration of the loop
  - Implied by loop precondition
  - Implies the loop postcondition
  - Implies next loop iteration is correct
- I(k): $i = k \ \wedge ((out = i \ \wedge A[out] = x) \vee (\exists j > i \ \ A[j] = x))$
- Guard condition: condition in the while loop
  - G= “out <0”
- Loop is correct when:
  - precondition $\rightarrow$ I(0)
  - for all k, $\mathrm{G} \wedge \mathrm{I(k)} \rightarrow \mathrm{I(k+1)}$
  - If k_0 is the smallest number such that $\neg G$, then $\neg G \wedge I(k_0) \rightarrow$ postcondition
- **Termination**: proof that $\exists\ \mathrm{k}_0$ such that after k_0 iterations G becomes false

$\exists i \in \{0 \dots n-1\} \ \ A[i] = x \ \wedge$
$\wedge \ \ i = 0 \wedge out = -1$

while $out < 0$ **do**
  if $A[i] = \mathrm{x}$ **then**
    $out = i$
  $i = i+1$

$A[out] = x$

Proving the loop invariant

$\exists i \in \{0 \ldots n-1\}\ A[i] = x\ \wedge$
$\wedge\ i = 0 \wedge out = -1$

while *out* < *0* **do**
    if $A[i]$ = x **then**
        out = *i*
    i = *i+1*

$A[out] = x$

- By induction on i:
- Base case: I(0)
  - $\exists i \in \{0 \ldots n-1\}\ A[i] = x \wedge i = 0 \wedge$
    $\wedge\ out = -1$

    Implies I(0)

  - $i = 0\ \wedge ((out = 0\ \wedge A[out] = x) \vee (\exists j > i\ \ A[j] = x))$
- Assume I(k): $i = k\ \wedge ((out = i\ \wedge A[out] = x) \vee (\exists j > i\ \ A[j] = x))$
- Show: if G, then I(k+1): $i = k+1\ \wedge ((out = i\ \wedge A[out] = x) \vee (\exists j > i\ \ A[j] = x))$
  - i=k+1 because of "i=i+1" statement
  - If A[i]=x, then $(out = i\ \wedge A[out] = x)$ holds
  - Otherwise, $(\exists j > i\ \ A[j] = x)$ holds.

– Otherwise, if $\neg G$, postcondition holds:
  - in this case, $(out = i\ \wedge A[out] = x)$ should have been true in I(k), for i=k.
  - So A[out]=x

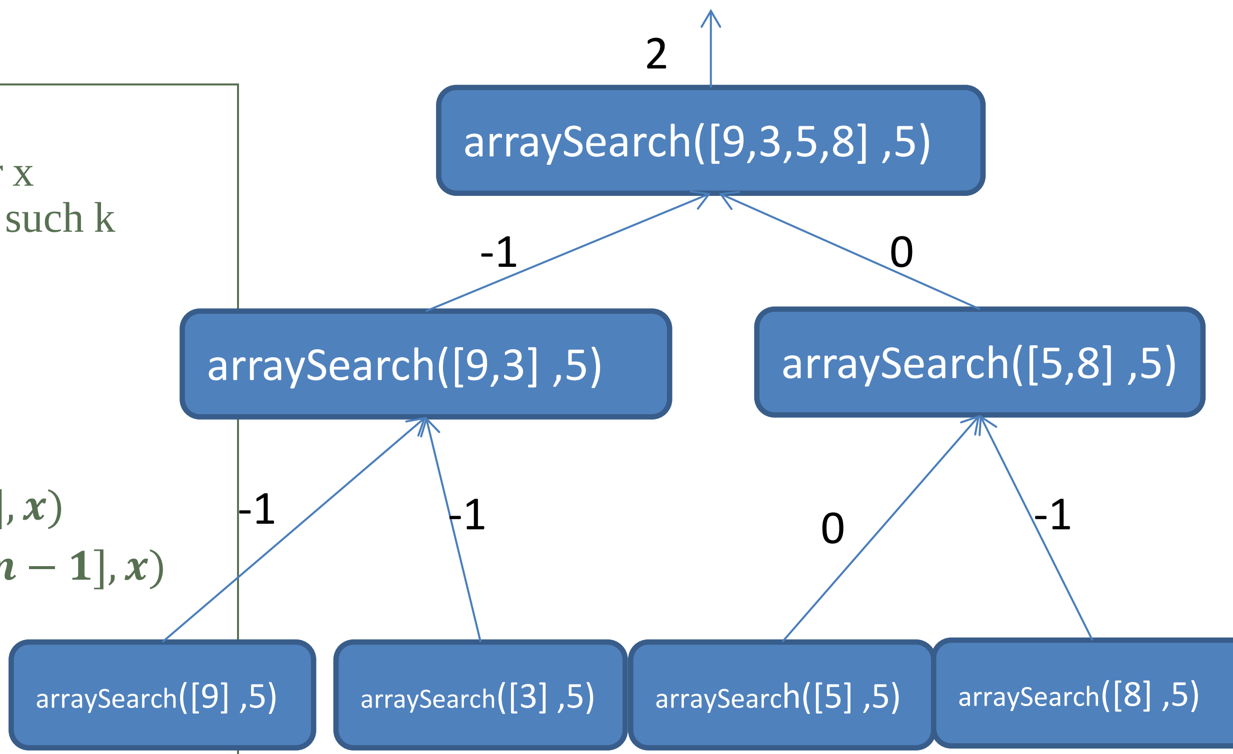
Correctness of recursive programs

Algorithm *arraySearch(A, x)*
Input array ***A*** of ***n*** integers, number x
Output k such that A[k]=x, -1 if no such k

if $A[0] = x$ **then**
  return 0

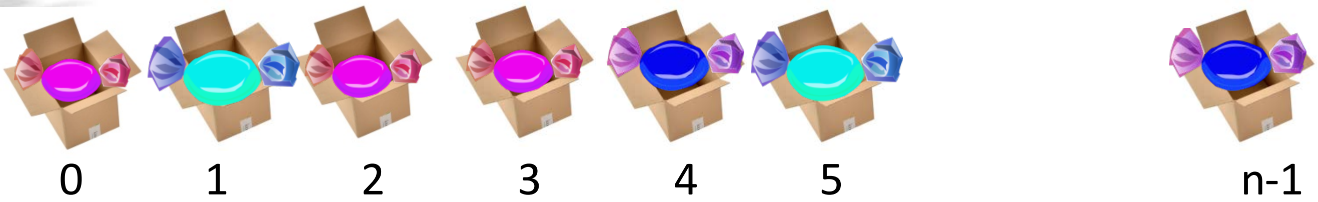
else if $n > 1$ **then**
  $first = arraySearch(A[0..\frac{n}{2} - 1], x)$
  $second = arraySearch(A[n/2,\ n - 1], x)$

  if $second > 0$ **then**
    return $second + n/2$
  else
    return *first*
else
  return *-1*



Use strong induction!
Assume both calls return correct value
Show that the program returns correct value

Running time: worst case

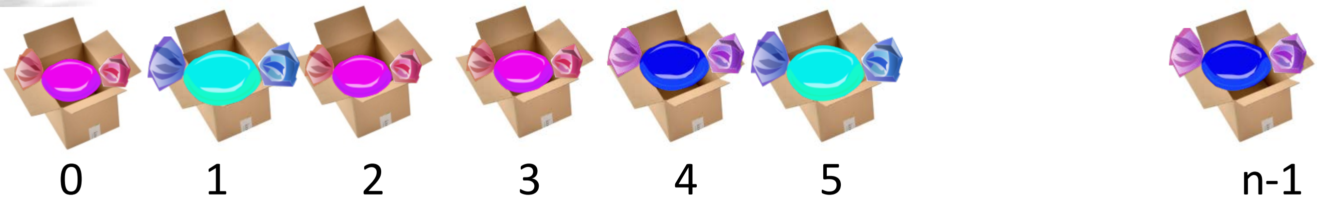


- *Precondition*: A is an array containing x
  - *Therefore, in the worst scenario need to check all n boxes A[i]*
  - *Running time: O(n)*

Algorithm *arraySearch(A, x)*
Input array ***A*** of ***n*** integers, number x
Output k such that A[k]=x

```
i = 0
out = -1
while  out < 0 do
        if A[i] = x then
                out = i
        i = i+1
return out
```

Running time: average case



- *What is the expected number of steps before x is found?*
  - *Depends on the probability of x being in each cell.*
  - *Or whether there is only one x, or can be many*

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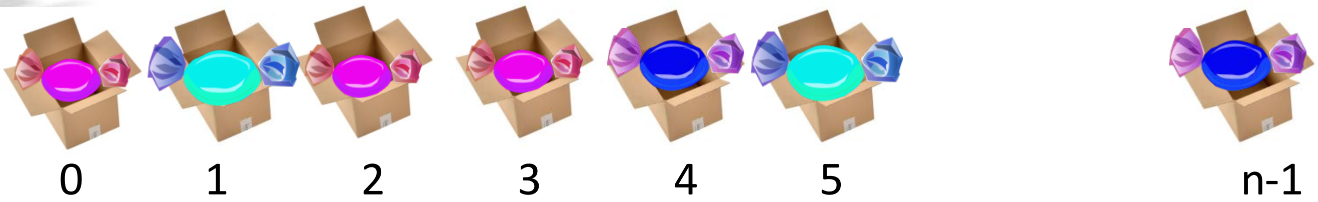
Bernoulli trials and repeated experiments

- Suppose an experiment has two outcomes, 1 and 0 (success/failure), with Pr(1) = p.
  - Such experiment is called a **Bernoulli trial**.
- What happens if the experiment is repeated multiple times (independently from each other?)

  - A sample space after carrying out n Bernoulli trials is a set of all possible n-tuples of elements in {0,1} (or {success, fail}).
  - Number of n-tuples with k 1s is $\binom{n}{k}$
  - Probability of getting 1 in any given trial is p, of getting 0 is (1-p).
  - Probability of getting exactly k 1s (successes) out of n trials is $\binom{n}{k}p^k(1-p)^{n-k}$
    - Called binomial distribution
  - Probability of getting the first success on exactly the k^{th} trial is $p(1-p)^{k-1}$

- How many trials do we need, on average, to get a success?

Running time: average case



- *Suppose probability of x being in any cell is p*
  - *Can have many x in A*
- *Then probability of finding x in k steps is* $p(1-p)^{k-1}$
- *Let random variable X denote the number of loop iterations till x is found*
- *E(X) =* $\Sigma_{i \in \mathbb{N}}\, i * \Pr(X = i) = \frac{1}{p}$
- *Expect to find x in O(1/p) steps*

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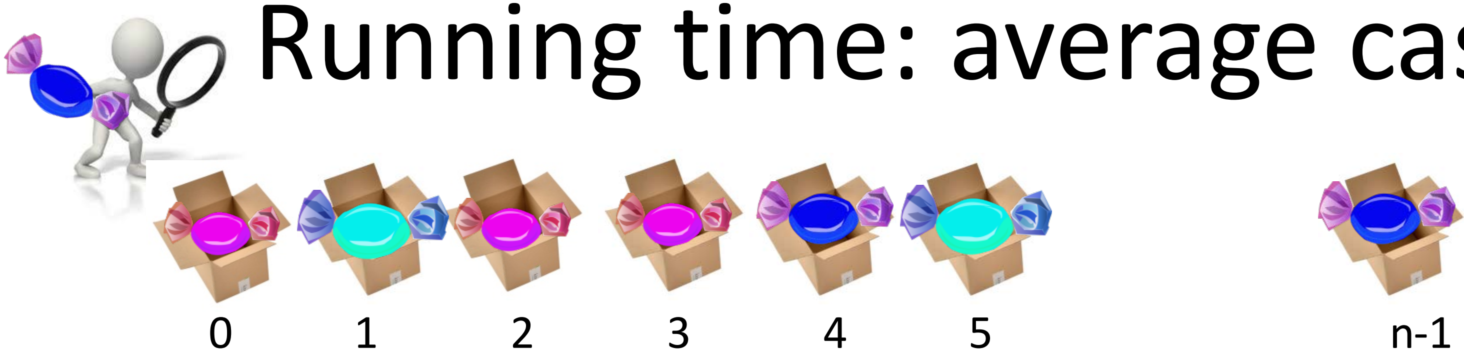
Running time: average case



- *Suppose there is just one x in A*
- *Probability of finding x in each step is* $\frac{1}{n}$

- *Let random variable X denote the number of loop iterations till x is found*
- $E(X) = \Sigma_{i=n}\, i * \Pr(X = i) = \frac{1}{n}\Sigma_{i=1}^{n} i$
  $= (n + 1)/2$

- *Expect to find x in the middle of A*
- *Running time O(n)*

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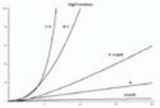
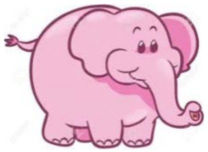
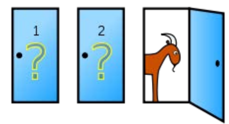



More to come…






- You will see a lot of algorithm analysis and use of the concepts we developed in COMP 2002 and beyond.
  - **Logic, sets, relations and graphs** for specification, modeling problems and describing what you are doing.
  - **Logic, induction and models of computation** for proving program correctness and analysis of problem complexity.
  - **Recursive definitions of algorithms, counting and probability** for algorithm performance and problem solving.
    - With the million dollar problem rearing its head every now and then











Have fun!