COMP 1002

Logic for Computer Scientists

Lecture 28
Finite probability

• More common: use the language of probability.

• **Experiments**: producing an **outcome** out of possible choices
  – Tossing a coin: outcome can be “heads”
  – Getting a lottery ticket: outcome can be “win”

• **Sample space** $S$: set of all possible outcomes.
  – $\{\text{heads, tails}\}$ for coin toss
  – $\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$ for rolling two dice

• **Event** $A \subseteq S$: subset of outcomes
  – Both dice came up even.

• **Probability** of an event if all outcomes are **equally likely**:
  – $Pr(A) = \frac{|A|}{|S|}$ (fraction of the outcomes that are in the event $A$).
  – Probability of both dice coming up even:
    • $A = \{(2,2),(2,4),(4,2),(2,6),(6,2),(4,4),(4,6),(6,4),(6,6)\}$. $|A| = 9$, $|S| = 36$
    • $P(A) = \frac{9}{36} = \frac{1}{4}$

• Can use the same combinatorics we just studied to calculate probabilities (i.e., for finding the size of $A$).
Probabilities and pink elephants

• What is the probability that walking down George street you’d see a pink elephant?
  – Your friend says: “It is ½! You will either see the pink elephant, or not!”
  • Do you agree?
Probabilities and distributions

• What if outcomes are not equally likely?
  – Biased coins, etc.

• A function \( Pr: S \to \mathbb{R} \) is a **probability distribution** on (a finite set) \( S \) if \( Pr \) satisfies the following:
  – For any outcome \( s \in S \), \( 0 \leq Pr(s) \leq 1 \)
  – \( \sum_{s \in S} Pr(s) = 1 \)

• **Uniform distribution**: for all \( s \in S \), \( Pr(s) = 1/|S| \)
  – all outcomes are equally likely
  – Fair coin: \( Pr(\text{heads}) = Pr(\text{tails}) = \frac{1}{2} \)

• Biased coin: say heads twice as likely as tails.
  – \( Pr(\text{heads}) + Pr(\text{tails}) = 1. \) \( Pr(\text{heads}) = 2 \times Pr(\text{tails}) \)
  – So \( Pr(\text{heads}) = \frac{2}{3}, \) \( Pr(\text{tails}) = \frac{1}{3} \)
Puzzle: Monty Hall problem

• Let’s make a deal!
  – A player picks a door.
  – Behind one door is a car.
  – Behind two others are goats.

• A player chooses a door.
  – A host opens another door
  – Shows a goat behind it.
  – And asks the player if she wants to change her choice.

• Should she switch?
Probabilities of events

• Probability of an event A is a sum of probabilities of the outcomes in A:
  \[ \Pr(A) = \sum_{a \in A} \Pr(a) \]

• Probability of A not occurring:
  \[ \Pr(\overline{A}) = 1 - \Pr(A) \]

• Probability of the union of two events (either A or B happens) is
  \[ \Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) \]
  • By principle of inclusion-exclusion
  \[ \text{If A and B are disjoint, then } \Pr(A \cap B) = 0, \text{ so } \Pr(A \cup B) = \Pr(A) + \Pr(B) \]

• In general, if events \( A_1 \ldots A_n \) are pairwise disjoint
  • that is, \( \forall i, j \text{ if } i \neq j \text{ then } A_i \cap A_j = \emptyset \)
  \[ \Pr(\bigcup_{i=1}^{n} A_i) = \Pr(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} \Pr(A_i) \]
  • That is, probability of that any of the events happens is the sum of their individual probabilities.
Probabilities of events

- Suppose a die is biased so that 3 appears twice as often as any other number (others equally likely).
  - Probability of 3: 2/7. Probabilities of others: 1/7

- What is the probability that an odd number appears?
  - Event: $A=\{1,3,5\}$
  - $Pr(A) = 1/7+2/7+1/7=4/7$.

- What is a probability that either an odd number or a number divisible by 3 appears?
  - $A=\{1,3,5\}$. $B = \{3,6\}$ . $A \cap B = \{3\}$
  - $Pr(A) = 4/7$. $Pr(B) = 3/7$. $Pr(A \cap B) = 2/7$
  - $Pr(A \cup B) = Pr(\{1,3,5,6\})$
    $$\frac{4}{7} + \frac{3}{7} - \frac{2}{7} = \frac{1}{7} + \frac{2}{7} + \frac{1}{7} + \frac{1}{7} = \frac{5}{7}$$
Bernoulli trials

• Bernoulli trials: repeatedly do an experiment with 2 outcomes ("success" and "failure")
  • Flip a coin, success if it comes head. Repeat.
  • Roll a pair of dice, success if numbers sum to 5. Repeat.
  – results of each repeat do not depend on previous
  – each experiment has probability $p$ of success
    • so probability $(1 - p)$ of failure.

• Probability of exactly $k$ successes out of $n$ Bernoulli trials is $\binom{n}{k} p^k (1 - p)^{n-k}$
  – This is called the "binomial distribution"
    • As a function of $k$, given $n$ and $p$. 
Birthday paradox

• How many people have to be in the room so that probability that two of them have the same birthday is at least \( \frac{1}{2} \)?
Birthday paradox

• How many people have to be in the room so that probability that two of them have the same birthday is at least ½?
  – Considering all birthdays independent: no twins!
  – And considering all days equally likely
    • Otherwise probability would be higher.
  – Even counting leap years: 366 days.

• Product rule: number of combinations of distinct birthdays of the first $i$ people is $P(i, 366) = 366\times365\times\ldots\times(366-i+1)$
  – Probability that the first $i$ people all have different birthday is $\frac{P(i,366)}{366^i} = \frac{365}{366} \times \frac{364}{366} \times \ldots \times \frac{366-i+1}{366}$
  – So with probability $1 - \frac{P(i,366)}{366^i}$ at least two out of first $i$ people have birthday on the same day.
  – That comes up to about $i = 23$ people to reach ½.
Let’s make a deal!
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- Behind one door is a car.
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A player chooses a door.
- A host opens another door
- Shows a goat behind it.
- And asks the player if she wants to change her choice.

Should she switch?
- Originally, probability of picking the car is 1/3
- If she first picked a door with a car: (1/3 probability)
  - Then she would switch to a goat.
- If she first picked a door with a goat (2/3 probability)
  - Then she would switch to a car.
### Conditional probabilities

- **Conditional probability** of an event $A$ given event $B$, denoted $\Pr(A|B)$, is the probability of $A$ if we know that $B$ occurred.
  - Probability of car $a$ behind door 2 if we chose door 1, and door 3 had a goat behind it.

- So it is probability of both $A$ and $B$, given that we know $B$ happened for sure:

\[
\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}
\]
  - Assume that $\Pr(B) > 0$: after all, $B$ did happen.
Independent events

• If knowing B gives us no information about A and vice versa, then A and B are **independent** events:
  
  Then $\Pr(A) = \Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$.

  – A and B are independent iff $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$.

• In general, events $A_1 \ldots A_n$ can be **pairwise independent** (that is, any two $A_i, A_j$) are independent, or (stronger condition) **mutually independent**: $\Pr(\cap_{i=1}^{n} A_i) = \Pi_{i=1}^{n} \Pr(A_i)$

  – For example, different coin tosses/dice rolls are usually considered independent.

• **Conditional independence**:
  
  – Conditioned on C happening, A and B are independent
  
  $\Pr(A \cap B \mid C) = \Pr(A \mid C) \cdot \Pr(B \mid C)$

  • Conditional independence does not imply independence, and vice versa.
Hat-check problem

• Suppose \( n \) men came to an event, and checked in their hats at the door.
  – On the way out, in a hurry, they each picked up a random hat.
  – On average, how many men picked their own hat?