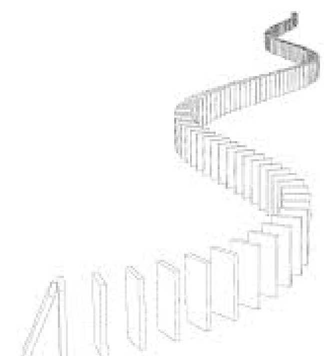
COMP 1002

Logic for Computer Scientists

Lecture 28

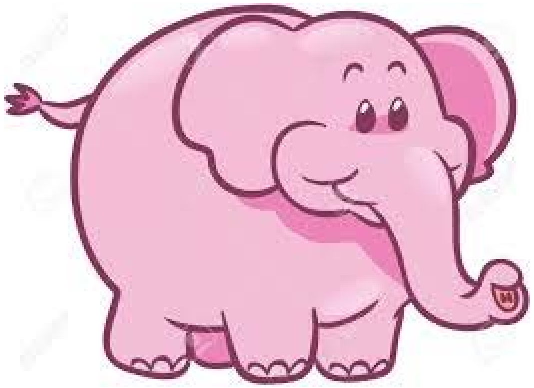


Finite probability

- More common: use the language of probability.
- **Experiments**: producing an **outcome** out of possible choices
  - Tossing a coin: outcome can be "heads"
  - Getting a lottery ticket: outcome can be "win"
- **Sample space S**: set of all possible outcomes.
  - {heads, tails} for coin toss
  - {1,2,3,4,5,6} x {1,2,3,4,5,6} for rolling two dice
- **Event A** $\subseteq S$: subset of outcomes
  - Both dice came up even.
- **Probability** of an event if all outcomes are **equally likely**:
  - Pr(A)= |A|/|S| (fraction of the outcomes that are in the event A).
  - Probability of both dice coming up even:
    - A={ (2,2),(2,4),(4,2),(2,6),(6,2),(4,4), (4,6), (6,4) , (6,6)}. |A| =9, |S|=36
    - P(A)=9/36=1/4
- Can use the same combinatorics we just studied to calculate probabilities (i.e., for finding the size of A).

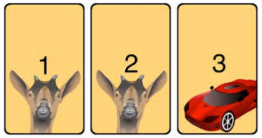
Probabilities and pink elephants

- What is the probability that walking down George street you'd see a pink elephant?
  - Your friend says: "It is ½! You will either see the pink elephant, or not!"
    - Do you agree?



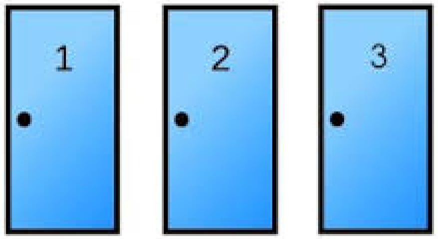
Probabilities and distributions

- What if outcomes are not equally likely?
  - Biased coins, etc.
- A function $Pr: S \rightarrow \mathbb{R}$ is a **probability distribution** on (a finite set) S if Pr satisfies the following:
  - For any outcome s ∈ S, $0 \leq \Pr(s) \leq 1$
  - $\Sigma_{\{s \in S\}} \Pr(s) = 1$
- **Uniform distribution**: for all $s \in S, \Pr(s) = 1/|S|$
  - all outcomes are equally likely
  - Fair coin: $\Pr(heads) = \Pr(tails) = \frac{1}{2}$
- Biased coin: say heads twice as likely as tails.
  - $\Pr(heads) + \Pr(tails) = 1.\ \Pr(heads) = 2 * \Pr(tails)$
  - So $\Pr(heads) = \frac{2}{3}, \Pr(tails) = \frac{1}{3}$

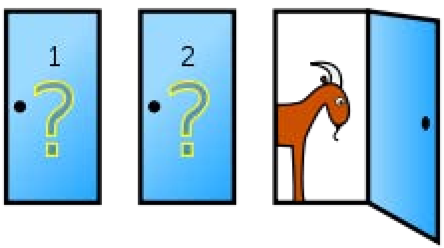


Puzzle: Monty Hall problem

- Let’s make a deal!
  - A player picks a door.
  - Behind one door is a car.
  - Behind two others are goats.



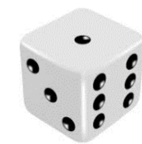
- A player chooses a door.
  - A host opens another door
  - Shows a goat behind it.
  - And asks the player if she wants to change her choice.



- Should she switch?

Probabilities of events

- Probability of an event A is a sum of probabilities of the outcomes in A:
  - Pr(A)=$\Sigma_{\{a \in A\}} \Pr(a)$



- Probability of A not occurring:
  - $\Pr(\bar{A}) = 1 - \Pr(A)$

- Probability of the union of two events (either A or B happens) is

$$\Pr(A \cup \mathrm{B}) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

    - By principle of inclusion-exclusion
  - If A and B are disjoint, then $\Pr(A \cap B) = 0$, so $\Pr(A \cup B) = \Pr(A) + \Pr(B)$

- In general, if events $A_1 \ldots A_n$ are pairwise disjoint
    - that is, $\forall i, j$ if $i \neq j$ then $A_i \cap A_j = \emptyset$
  - Then Pr($\cup_{i=1}^{n} A_i) = \Pr(A_1 \cup A_2 \cup \cdots \cup A_n) = \Sigma_{i=1}^{n} \Pr(A_i)$
    - That is, probability of that any of the events happens is the sum of their individual probabilities.

Probabilities of events

- Suppose a die is biased so that 3 appears twice as often as any other number (others equally likely).
  - Probability of 3: 2/7. Probabilities of others: 1/7

- What is the probability that an odd number appears?
  - Event: A={1,3,5}
  - Pr(A) = 1/7+2/7+1/7=4/7.

- What is a probability that either an odd number or a number divisible by 3 appears?
  - A={1,3,5}. B = {3,6} . $A \cap B = \{3\}$
  - Pr(A) = 4/7. Pr(B) = 3/7. $\Pr(A \cap B) = 2/7$
  - $\Pr(A \cup B) = \Pr(\{1,3,5,6\})$

$$= \frac{4}{7} + \frac{3}{7} - \frac{2}{7} = \frac{1}{7} + \frac{2}{7} + \frac{1}{7} + \frac{1}{7} = \frac{5}{7}$$

Bernoulli trials

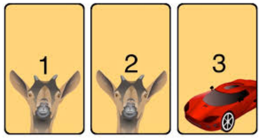
- Bernoulli trials: repeatedly do an experiment with 2 outcomes (“success” and “failure”)
    - Flip a coin, success if it comes head. Repeat.
    - Roll a pair of dice, success if numbers sum to 5. Repeat.
  - – results of each repeat do not depend on previous
  - – each experiment has probability p of success
    - so probability $(1-p)$ of failure.
- Probability of exactly k successes out of n Bernoulli trials is $\binom{n}{k}p^k(1-p)^{n-k}$
  - – This is called the “binomial distribution”
    - As a function of k, given n and p.

Birthday paradox

- How many people have to be in the room so that probability that two of them have the same birthday is at least ½?


Birthday paradox

- How many people have to be in the room so that probability that two of them have the same birthday is at least ½?
  - Considering all birthdays independent: no twins!
  - And considering all days equally likely
    - Otherwise probability would be higher.
  - Even counting leap years: 366 days.
- Product rule: number of combinations of distinct birthdays of the first i people is P(i, 366) = 366*365*…*(366-i+1)
  - Probability that the first i people all have different birthday is $\frac{P(i,366)}{366^i} = \frac{365}{366}\frac{364}{366}\dots\frac{(366-i+1)}{366}$
  - So with probability $1 - \frac{P(i,366)}{366^i}$ at least two out of first i people have birthday on the same day.
  - That comes up to about i =23 people to reach ½.



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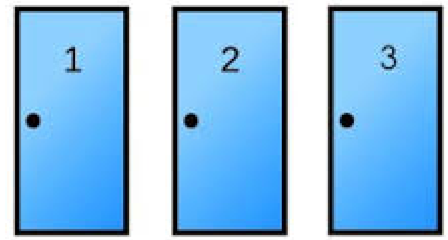

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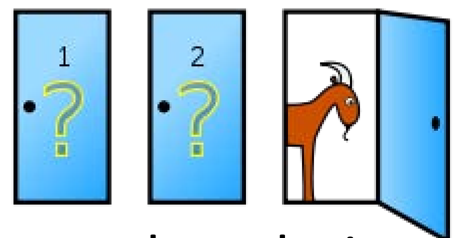

- A player chooses a door.
  - A host opens another door
  - Shows a goat behind it.
  - And asks the player if she wants to change her choice.
- Should she switch?
  - Originally, probability of picking the car is 1/3
  - If she first picked a door with a car: (1/3 probability)
    - Then she would switch to a goat.
  - If she first picked a door with a goat (2/3 probability)
    - Then she would switch to a car.

- **Conditional probability** of an event A given event B, denoted $\Pr(A|B)$, is the probability of A if we know that B occurred.
  - Probability of car a behind door 2 if we chose door 1, and door 3 had a goat behind it.
- So it is probability of both A and B, given that we know B happened for sure:

$$\Pr(A|B) = \frac{\Pr(A \cap B\ )}{\Pr(B)}$$

  - Assume that $\Pr(B) > \ 0$: after all, B did happen.

Conditional probabilities

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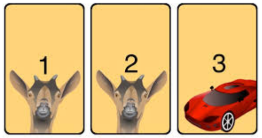

Independent events

- If knowing B gives us no information about A and vice versa, then A and B are **independent** events:
  - Then Pr(A) = Pr(A|B) = $\frac{\Pr(A \cap B)}{\Pr(B)}$.
  - A and B are independent iff $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$.
    - In general, events $A_1 \ldots A_n$ can be **pairwise independent** (that is, any two A_i, A_j) are independent, or (stronger condition) **mutually independent**: $\Pr(\cap_{i=1}^{n} A_i) = \Pi_{i=1}^{n} \Pr(A_i)$
  - For example, different coin tosses/dice rolls are usually considered independent.

- Conditional independence:
  - Conditioned on C happening, A and B are independent
  - $\Pr(A \cap B \mid C) = \Pr(A|C) \cdot \Pr(B|C)$
    - Conditional independence does not imply independence, and vice versa.

Hat-check problem

- Suppose n men came to an event, and checked in their hats at the door.
  - On the way out, in a hurry, they each picked up a random hat.
  - On average, how many men picked their own hat?