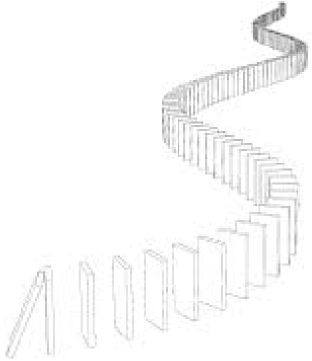
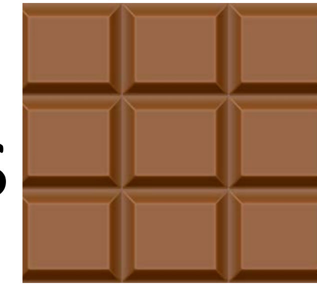
COMP 1002

Logic for Computer Scientists

Lecture 26



Puzzle: chocolate squares



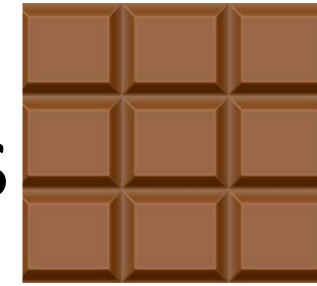
- Suppose you have a piece of chocolate like this:

- How many squares are in it?
  - of all sizes,
  - from single to the whole thing

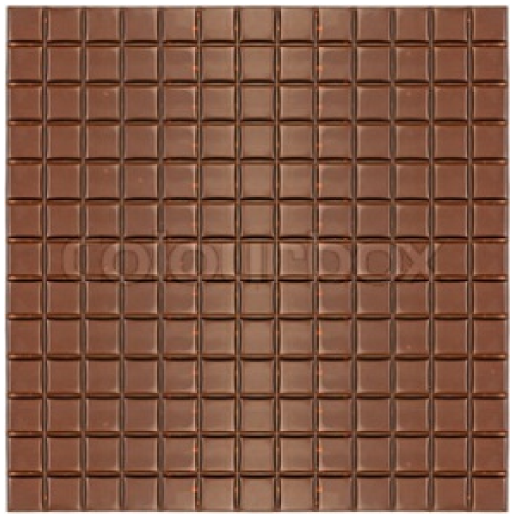
1. One square 4x4
2. Four squares 3x3
   - Can start (e.g, top-left corner) at (1,1), (1,2), (2,1), (2,2)
   - 2 choices for a row, 2 choices for a column.
3. Nine squares 2x2
   - Can start at any (x,y) with $x \in \{1,2,3\}$, $y \in \{1,2,3\}$
   - So 3*3 = 9 choices.
4. 16 squares 1x1.

- Total: 1+4+9+16=30 squares

Puzzle: chocolate squares



- Suppose you have a piece of chocolate like this:



- How many squares are in it?
  – of all sizes,
  – from single to the whole thing

1. One square 13x13
2. …
3. 13*13=169 squares 1x1
   - 12*12 2x2
   - 11*11 3x3
   - ….

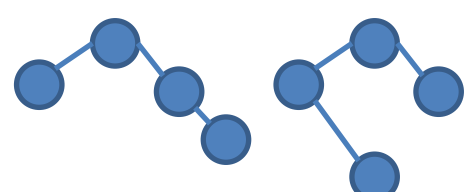
General formula: starting with an n x n piece of chocolate, get

$\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$

Total number of squares.

Combinatorics

- Counting various ways to arrange things.
  - Leading to probability theory.
- How long would a program that does brute-force search over possibilities will run?
  - Depends on the number of potential answers.
    - How many ways a sorting algorithm can rearrange its n-element input?
- How many possibilities need to be checked to break 4-digit PIN? A 8-letter password (with digits and special symbols)? A 80-letter passphrase (with just letters)? Which one is more secure?
    - 10 digits + 23 special symbols + 26 lowercase + 26 uppercase letters.
- What if some of the possibilities are identical to each other? How can we count then?
  - How many different trees are there, if two trees are considered the same if one can be transformed into another by moving/renaming vertices, keeping edges attached.
    - Flip and move bottom vertex.



Rules of sum and product

- Rule of sum:
  - If there are n choices for A, and m choices for B, then there are n+m choices for "A or B"
    - Provided A and B do not overlap.
    - If there are 16 squares of size 1, and 9 squares of size 2, then there are 25 squares of size either 1 or 2.
- Rule of product:
  - If there are n choices for A, and m choices for B, then there are n*m choices for "A and B".
    - 3 choices for a row times 3 choices for a column: 9 of 2x2 squares.
      - Can also count rectangles rather than squares…

Cartesian products

- When a sequence is a Cartesian product of n copies of the same set:
  - How many possible PINs consisting of 4 digits are there?
    - Using the rule of product. Each of the digits has 10 possibilities (0…9), and picking a digit for one position does not affect others. $10*10*10*10=10^4 = 10{,}000$.
    - So by the Pigeon Hole Principle, there are (lots of) people at MUN that have the same PIN.
  - How many rows does a truth table on n variables have?
    - Each variable has 2 possibilities. So 2^n.
  - In general, if there are n independent places in the sequence, and m possibilities for each place, get m^n possible sequences.

- When a sequence is a Cartesian product of different sets:
  - Multiply together sizes of these sets.
  - How many different dishes can you make out of: 3 types of protein (meat, chicken, falafel), 2 types of starch (noodles, rice), 4 types of sauces?
    - 3*2*4.

Permutations

- **Permutations**: number of sequences of objects.
  - Without repetition: each object appears once.
- Example: how many ways to assign offices ER-6030 to ER-6033 to Antonina, Dave, Sharene and Yuanzhu?
  - 4 choices to pick who gets ER-6030. This leaves 3 choices to pick who gets ER-6031. Now 2 remain for ER-6032, and the last one is stuck with ER-6033.
    - By the product rule, get 4*3*2*1 = 4!
- In general, number of permutations of n elements is n!
  - "Permutations": the difference between choices is only the order of elements.

r-Permutations.

- Before, we talked about permutations of all objects in the set. An **r-permutation** P(n,r) involves taking only r out of n objects, and counting the number of possible sequences.
  - That is, the task consists of 1) picking r out of n objects. 2) counting number of sequences out of them.
- Example:
  - How many ways to assign ER offices to 4 out of 25 faculty members?
    - Pick one out of 25 to be in ER-6030. One out of remaining 24 to be in ER-6031, one out of 23 for ER-6032, one out of 22 for ER-6033. By the product rule, get P(25,4) = 25*24*23*22.
  - Alternative way of thinking:
    - There are 25! ways to assign offices to everybody.
    - Out of them, 21! way to assign non-ER offices. We are not interested in what everybody else got – so once the way to assign ER offices is fixed, all sequences with this assignment to ER offices are the same for us.
    - Get $\mathrm{P}(25,4) = \frac{25!}{21!}$ =25*24*23*22 possible ways to get 4 people offices in ER.
- General formula: $P(n,r) = \frac{n!}{(n-r)!}$

Combinations.

- In general, how many ways to pick k out of n objects?
  - The number of k-permutations divided by the number of different permutations of the k objects themselves.
- **Combinations**: C(n,k) is number of ways to choose k objects out of n.
  - "n choose k": $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ = C(n,k)

  - How many ways to choose 4 out of 25 faculty members to get an office in Earth Science?
    - 25! ways to assign offices altogether.
    - 4! ways to assign the first 4 offices (ones in Earth Science).
    - 21! ways to assign offices not in Earth Science.
    - Overall, C(25,4)=$\frac{25!}{4!\ 21!}$ ways to pick who gets an Earth Science office.

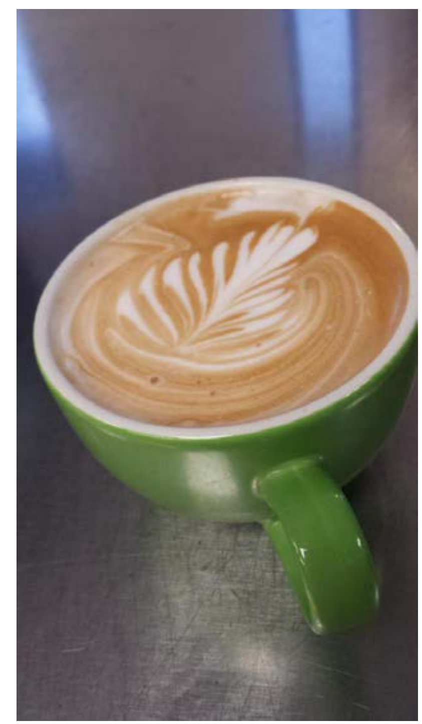
  - How many ways to select a crew of 6 astronauts out of a team of 30 to go to Mars?
    - C(30,6) = $\binom{30}{6} = \frac{30!}{6!(24!)} = 593{,}775$

Permanent link to this comic: https://xkcd.com/1609/

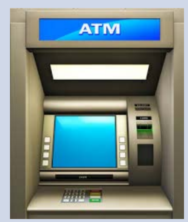
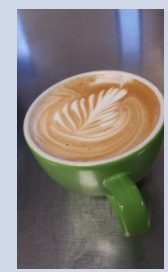
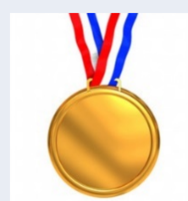
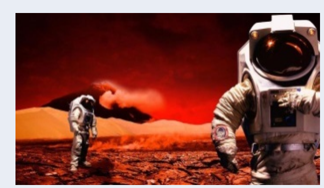
Combinations with repetition

- Suppose that 10 of you came to the office hour, and we decided to go to Jumping Bean to get some coffee.
- Jumping Bean sells 6 types of coffee drinks: drip coffee, cappucino, espresso, latte, mocca and americano.
- How many different orders can we place, if each gets one coffee drink?
  - That is, how many ways are there to select 11 items, where each item comes from one of the 6 categories?
    - Let's use one letter for each type:
    - e.g., aacdeeellmm stands for 2 americanos, one cappucino, one drip, three espressos, 2 lattes and 2 moccas.
  - Idea: think of a string of letters with "dividers" between different types of drinks:
    - aa|c|d|eee|ll|mm
  - How many ways are there to position the dividers?
    - 6-1= 5 dividers
    - Number of orders + number of dividers: 11+6-1=16 positions.
    - So $\binom{16}{11} = \binom{16}{5}$ = 4368 possible orders.

- In general, number of ways to select r objects out of n categories with repetition is $\binom{r+n-1}{r}$



Summary

Selecting k out of n objects	Order matters (permutations)	Order ignored (combinations)
With repetitions	n^k 	$\binom{k+n-1}{k}$ 
Without repetitions	$P(n,k) = \frac{n!}{(n-k)!}$ 	$\binom{n}{k}$ 

Puzzle: misspelling OSOYOOS

- In the game of Scrabble, players make words out of the pieces they have.
  - Suppose that someone puts the word "OSOYOOS" on the board, using up all her pieces.
  - How many ways could she have had the letters arranged on the rack in front of them?
    - The order of multiple copies of a letter does not matter: switching two S around results in the same sequence, but switching O and S does not.
    - The letters on the rack do not have to form a word.