Puzzle: chocolate squares

• Suppose you have a piece of chocolate like this:

• How many squares are in it?
  – of all sizes,
  – from single to the whole thing

1. One square 4x4
2. Four squares 3x3
   • Can start (e.g, top-left corner) at (1,1), (1,2), (2,1), (2,2)
   • 2 choices for a row, 2 choices for a column.
3. Nine squares 2x2
   • Can start at any (x,y) with \( x \in \{1,2,3\}, y \in \{1,2,3\}\)
   • So 3*3 = 9 choices.
4. 16 squares 1x1.

• Total: 1+4+9+16=30 squares
Puzzle: chocolate squares

• Suppose you have a piece of chocolate like this:

• How many squares are in it?
  – of all sizes,
  – from single to the whole thing

1. One square 13x13
2. ...
3. 13*13=169 squares 1x1
   • 12*12 2x2
   • 11*11 3x3
   • ....

General formula: starting with an $n \times n$ piece of chocolate, get

$$\sum_{i=1}^{n} i^2 = n(n + 1)(2n + 1)/6$$

Total number of squares.
Combinatorics

• Counting various ways to arrange things.
  – Leading to probability theory.

• How long would a program that does brute-force search over possibilities will run?
  – Depends on the number of potential answers.
    • How many ways a sorting algorithm can rearrange its n-element input?

• How many possibilities need to be checked to break 4-digit PIN? A 8-letter password (with digits and special symbols)? A 80-letter passphrase (with just letters)? Which one is more secure?
  • 10 digits + 23 special symbols + 26 lowercase + 26 uppercase letters.

• What if some of the possibilities are identical to each other? How can we count then?
  – How many different trees are there, if two trees are considered the same if one can be transformed into another by moving/renaming vertices, keeping edges attached.
    • Flip and move bottom vertex.
Rules of sum and product

• Rule of sum:
  – If there are \( n \) choices for A, and \( m \) choices for B, then there are \( n+m \) choices for “A or B”
    • Provided A and B do not overlap.
    • If there are 16 squares of size 1, and 9 squares of size 2, then there are 25 squares of size either 1 or 2.

• Rule of product:
  – If there are \( n \) choices for A, and \( m \) choices for B, then there are \( n*m \) choices for “A and B”.
    • 3 choices for a row times 3 choices for a column: 9 of 2x2 squares.
      – Can also count rectangles rather than squares...
Cartesian products

• When a sequence is a Cartesian product of n copies of the same set:
  – How many possible PINs consisting of 4 digits are there?
    • Using the rule of product. Each of the digits has 10 possibilities (0...9), and picking a digit for one position does not affect others.
      \[10 \times 10 \times 10 \times 10 = 10^4 = 10,000.\]
    • So by the Pigeon Hole Principle, there are (lots of) people at MUN that have the same PIN.
  – How many rows does a truth table on n variables have?
    • Each variable has 2 possibilities. So \(2^n\).
  – In general, if there are n independent places in the sequence, and m possibilities for each place, get \(m^n\) possible sequences.

• When a sequence is a Cartesian product of different sets:
  – Multiply together sizes of these sets.
  – How many different dishes can you make out of: 3 types of protein (meat, chicken, falafel), 2 types of starch (noodles, rice), 4 types of sauces?
    • 3*2*4.
Permutations

- **Permutations**: number of sequences of objects.
  - Without repetition: each object appears once.
- Example: how many ways to assign offices ER-6030 to ER-6033 to Antonina, Dave, Sharene and Yuanzhu?
  - 4 choices to pick who gets ER-6030. This leaves 3 choices to pick who gets ER-6031. Now 2 remain for ER-6032, and the last one is stuck with ER-6033.
    - By the product rule, get $4 \times 3 \times 2 \times 1 = 4!$
- In general, number of permutations of $n$ elements is $n!$
  - “Permutations”: the difference between choices is only the order of elements.
r-Permutations.

- Before, we talked about permutations of all objects in the set. An \textbf{r-permutation} \( P(n,r) \) involves taking only \( r \) out of \( n \) objects, and counting the number of possible sequences.
  - That is, the task consists of 1) picking \( r \) out of \( n \) objects. 2) counting number of sequences out of them.

- Example:
  - How many ways to assign ER offices to 4 out of 25 faculty members?
    - Pick one out of 25 to be in ER-6030. One out of remaining 24 to be in ER-6031, one out of 23 for ER-6032, one out of 22 for ER-6033. By the product rule, get \( P(25,4) = 25\times24\times23\times22 \).
    - Alternative way of thinking:
      - There are 25! ways to assign offices to everybody.
      - Out of them, 21! way to assign non-ER offices. We are not interested in what everybody else got – so once the way to assign ER offices is fixed, all sequences with this assignment to ER offices are the same for us.
      - Get \( P(25,4) = \frac{25!}{21!} = 25\times24\times23\times22 \) possible ways to get 4 people offices in ER.

- General formula: \( P(n,r) = \frac{n!}{(n-r)!} \)
Combinations.

• In general, how many ways to pick k out of n objects?
  – The number of k-permutations divided by the number of different permutations of the k objects themselves.

• **Combinations**: \( C(n,k) \) is number of ways to choose k objects out of n.
  – “n choose k”: \( \binom{n}{k} = \frac{n!}{k!(n-k)!} = C(n,k) \)
  – How many ways to choose 4 out of 25 faculty members to get an office in Earth Science?
    • 25! ways to assign offices altogether.
    • 4! ways to assign the first 4 offices (ones in Earth Science).
    • 21! ways to assign offices not in Earth Science.
    • Overall, \( C(25,4) = \frac{25!}{4! \cdot 21!} \) ways to pick who gets an Earth Science office.

• How many ways to select a crew of 6 astronauts out of a team of 30 to go to Mars?
  • \( C(30,6) = \binom{30}{6} = \frac{30!}{6! \cdot 24!} = 593,775 \)
ICE CREAM  HAM  RELISH
PANCAKES  KETCHUP  CHEESE
EGGS  CUPCAKES  SOUR CREAM
HOT CHOCOLATE  AVOCADO  SKITTLES

YOU KNOW WHAT'S ACTUALLY REALLY GOOD? FOOD AND FOOD.

HUH. I GUESS I CAN SEE IT.

FUN FACT: IF YOU SAY "YOU KNOW WHAT'S ACTUALLY REALLY GOOD?" IN THE RIGHT TONE OF VOICE, YOU CAN NAME ANY TWO INDIVIDUALLY-GOOD FOODS HERE AND NO ONE WILL CHALLENGE YOU ON IT.
Suppose that 10 of you came to the office hour, and we decided to go to Jumping Bean to get some coffee.

Jumping Bean sells 6 types of coffee drinks: drip coffee, cappucino, espresso, latte, mocca, and americano.

How many different orders can we place, if each gets one coffee drink?

- That is, how many ways are there to select 11 items, where each item comes from one of the 6 categories?
  - Let’s use one letter for each type:
    - e.g., aacdeellmm stands for 2 americos, one cappucino, one drip, three espressos, 2 lattes and 2 moccas.
  - Idea: think of a string of letters with “dividers” between different types of drinks:
    - aa|c|d|eee|ll|mm
  - How many ways are there to position the dividers?
    - 6-1 = 5 dividers
    - Number of orders + number of dividers: 11+6-1=16 positions.
    - So \( \binom{16}{11} \) = \( \binom{16}{5} \) = 4368 possible orders.

In general, number of ways to select \( r \) objects out of \( n \) categories with repetition is \( \binom{r+n-1}{r} \).
## Summary

<table>
<thead>
<tr>
<th>Selecting k out of n objects</th>
<th>Order matters (permutations)</th>
<th>Order ignored (combinations)</th>
</tr>
</thead>
<tbody>
<tr>
<td>With repetitions</td>
<td>$n^k$</td>
<td>$\binom{k + n - 1}{k}$</td>
</tr>
<tr>
<td>Without repetitions</td>
<td>$P(n, k) = \frac{n!}{(n-k)!}$</td>
<td>$\binom{n}{k}$</td>
</tr>
</tbody>
</table>
Puzzle: misspelling OSOYOOS

• In the game of Scrabble, players make words out of the pieces they have.
  – Suppose that someone puts the word “OSOYOOS” on the board, using up all her pieces.
  – How many ways could she have had the letters arranged on the rack in front of them?
    • The order of multiple copies of a letter does not matter: switching two S around results in the same sequence, but switching O and S does not.
    • The letters on the rack do not have to form a word.