COMP 1002

Logic for Computer Scientists

Lecture 25
Structural induction

• Let $S \subseteq U$ be a recursively defined set, and $F(x)$ is a property (of $x \in U$).

• Then
  – if all $x$ in the base of $S$ have the property,
  – and applying the recursion rules preserves the property,
  – then all elements in $S$ have the property.
Multiples of 3

- Let’s define a set $S$ of numbers as follows.
  - Base: $3 \in S$
  - Recursion: if $x, y \in S$, then $x + y \in S$
- Claim: all numbers in $S$ are divisible by 3
  - That is, $\forall x \in S \exists z \in \mathbb{N} x = 3z$.
- Proof (by structural induction).
  - Base case: $3$ is divisible by 3 ($y=1$).
  - Recursion: Let $x, y \in S$. Then $\exists z, u \in \mathbb{N} x = 3z \land y = 3u$.
  - Then $x + y = 3z + 3u = 3(z + u)$.
  - Therefore, $x + y$ is divisible by 3.
  - As there are no other elements in $S$ except for those constructed from 3 by the recursion rule, all elements in $S$ are divisible by 3.
Binary trees

• **Rooted trees** are trees with a special vertex designated as a root.
  – Rooted trees are **binary** if every vertex has at most three edges: one going towards the root, and two going away from the root. Full if every vertex has either 2 or 0 edges going away from the root.

• Recursive definition of full binary trees:
  – Base: A single vertex \( v \) is a full binary tree with that vertex as a root.
  – Recursion:
    • Let \( T_1, T_2 \) be full binary trees with roots \( r_1, r_2 \), respectively. Let \( v \) be a new vertex.
    • A new full binary tree with root \( v \) is formed by connecting \( r_1 \) and \( r_2 \) to \( v \).
  – Restriction:
    • Anything that cannot be constructed with this rule from this base is not a full binary tree.
Height of a full binary tree

- **The height** of a rooted tree, $h(T)$, is the maximum number of edges to get from any vertex to the root.
  - Height of a tree with a single vertex is 0.
- **Claim:** Let $n(T)$ be the number of vertices in a full binary tree $T$. Then $n(T) \leq 2^{h(T)+1} - 1$
- **Proof (by structural induction)**
  - Base case: a tree with a single vertex has $n(T) = 1$ and $h(T) = 0$. So $2^{h(T)+1} - 1 = 1 \geq 1$
  - Recursion: Suppose $T$ was built by attaching $T_1, T_2$ to a new root vertex $v$.
    - Number of vertices in $T$ is $n(T) = n(T_1) + n(T_2) + 1$
    - Every vertex in $T_1$ or $T_2$ now has one extra step to get to the new root in $T$. So $h(T) = 1 + \max(h(T_1), h(T_2))$
    - By the induction hypothesis, $n(T_1) \leq 2^{h(T_1)+1} - 1$ and $n(T_2) \leq 2^{h(T_2)+1} - 1$
    - $n(T) = n(T_1) + n(T_2) + 1$
      \[ \leq 1 + (2^{h(T_1)+1} - 1) + (2^{h(T_2)+1} - 1) \]
      \[ \leq 2 \cdot \max(2^{h(T_1)+1}, 2^{h(T_2)+1}) - 1 \]
      \[ \leq 2 \cdot 2^{\max(h(T_1), h(T_2)) + 1} - 1 \]
      \[ = 2 \cdot 2^{h(T)} - 1 = 2^{h(T)+1} - 1 \]
  - Therefore, the number of vertices of any binary tree $T$ is less than $2^{h(T)+1} - 1$
- **Alternatively,** height of a binary tree is at least $\log_2 n(T)$
  - If you have a recursive program that calls itself twice (e.g, within if ... then ... else ...)
  - Then if this code executes $n$ times (maybe on $n$ different cases)
  - Then the program runs in time at least $\log_2 n$, even when cases are checked in parallel.