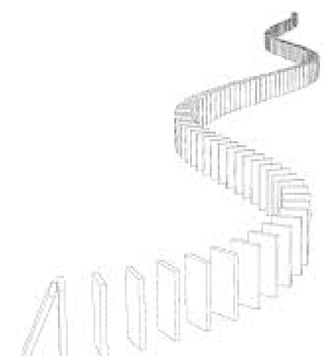
COMP 1002

Logic for Computer Scientists

Lecture 24



Deductive, Inductive, and Abductive Reasoning

- Deductive reasoning concerns what logically follows from given premises (if *a*, then *b*).
- Inductive reasoning—the process of deriving a reliable generalization from observations. Thus its validity requires us to define a reliable generalization method.
- Abductive reasoning that goes from observation to a hypothesis and seeks to find the simplest and most likely relevant evidence.

Mathematical induction

- Is a reliable generalization method.
- **Mathematical Induction principle:**

If $\mathrm{P}(0) \wedge \forall\, k \in \mathbb{N}$ P(k) → P(k+1) then

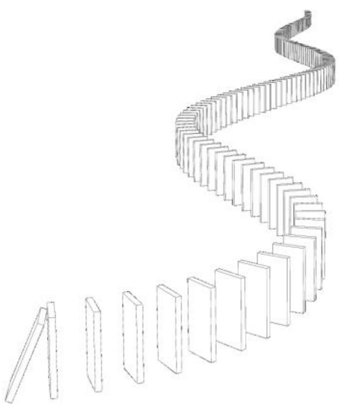
$\forall x \in \mathbb{N}\; P(x)$

Well-Order Principle

- (S, ≤) is a well-ordered set if it is a poset such that ≤ is a total ordering and every nonempty subset of S has a least element.
  - A relation R on a set S is reflexive, antisymmetric, and transitive, it is a partial ordering. A set S together with a partial ordering R is called partial ordering set, or poset.

Reliability of Mathematical induction

- Want to prove a statement $\forall x \in \mathbb{N}\ \ P(x)$.
  - – Check that $P(0)$ holds
  - – And whenever $P(k)$ does not hold for some k, $P(k-1)$ does not hold either
    - Contradicting well-ordering principle.
    - Contrapositive:
      - – if P(k-1) holds for arbitrary k,
      - – then P(k) also must be true.
  - – Conclude that $\forall x \in \mathbb{N}\ \ P(x)$



Mathematical induction

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Proving that P(0) holds
is called the **base case**.

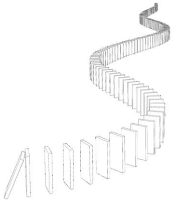
That P(k-1) holds is an **induction hypothesis**

Proving that P(k-1) → P(k)
Is the **induction step**

Mathematical Induction principle:
If $\mathrm{P}(0) \wedge \forall\, k \in \mathbb{N}$ P(k) → P(k+1) then $\forall x \in \mathbb{N}\ P(x)$

Sum of numbers formula

- Claim: for any n∈ $\mathbb{N}$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$
- Proof (by induction).
  - P(n) is $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ (*statement* we are proving by induction on n)
  - *Base case:* k=0. Then $\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$.
  - *Induction hypothesis:* Assume that $\sum_{i=0}^{k-1} i = \frac{(k-1)k}{2}$ for an arbitrary k >0
    - That is, for an arbitrary number k-1 ∈ $\mathbb{N}$
    - Can take k instead of k-1, but k-1 makes calculations simpler.
  - *Induction step:* show that P(k-1) implies P(k).
    - $\sum_{i=0}^{k} i = (\sum_{i=1}^{k-1} i)$ +k.
    - By induction hypothesis, $\sum_{i=1}^{k-1} i = \frac{(\mathrm{k}-1)\mathrm{k}}{2}$
    - Now, $\sum_{i=1}^{k} i = (\sum_{i=1}^{k-1} i)$ +k = $\frac{(\mathrm{k}-1)\mathrm{k}}{2} + \mathrm{k} = \frac{\mathrm{k}^2-\mathrm{k}+2\mathrm{k}}{2} = \frac{\mathrm{k}^2+k}{2} = \frac{k(k+1)}{2}$
  - *By induction*, therefore, P(n) holds for all $n \in \mathbb{N}$.



Changing the base case

- Mathematical Induction principle:
  - $(\mathrm{P}(0) \wedge \forall\, k \in \mathbb{N}\ \ \mathrm{P(k)} \rightarrow \mathrm{P(k+1)}) \rightarrow \forall x \in \mathbb{N}\ P(x)$

- What if want to prove it only for $x \geq a$?
  - Make a the base case (when $a \geq 0$). For the rest, assume $k \geq a$.
  - $(\mathrm{P(a)} \wedge \forall\, k \geq a\ \ \mathrm{P(k)} \rightarrow \mathrm{P(k+1)}) \rightarrow \forall x \geq a\ \ P(x)$
    - Here, $\forall x \geq a\ \ P(x)$ is a shorthand for $\forall x \in \mathbb{N}\ \ \big(x \geq a \rightarrow P(x)\big)$
  - To prove it works, prove P(n') where n'=n-a.

- Example: show that for all $n \geq 4,\ 2^n \geq n^2$
  - $P(n)$: $2^n \geq n^2$
  - Base case: n=4. $2^4 = 16 = 4^2$
  - Induction hypothesis: assume that for an arbitrary $k \geq a,\ 2^k \geq k^2$
  - Induction step: show that $2^k \geq k^2$ implies $2^{k+1} \geq (k+1)^2$
    - $2^{k+1} = 2 \cdot 2^k = 2^k + 2^k \geq k^2 + k^2$
    - $(k+1)^2 = k^2 + 2k + 1.$
    - Want: $k^2 + k^2 \geq k^2 + 2k + 1$, so $k^2 \geq 2k + 1$
      - Dividing both sides of the inequality by k: $k \geq 2 + \frac{1}{k}$
      - Since k ≥ 4, and $2 + \frac{1}{k} \leq 3$, $2 + \frac{1}{k} \leq 3 < 4 \leq k$. So $k \geq 2 + \frac{1}{k}$ and thus $k^2 \geq 2k + 1$
    - So $2^{k+1} = 2 \cdot 2^k = 2^k + 2^k \geq k^2 + k^2 \geq k^2 + 2k + 1 = (k+1)^2$
  - By induction, for all $n \geq 4,\ 2^n \geq n^2$
- Corollary: as n grows, an algorithm running in time n^2 will quickly outperform an algorithm running in time 2^n

Examples of mathematical induction

1. Find and prove closed forms
   1) $1/2, 4/5, 9/10, 16/17, \ldots$
   2) $1, 2/2, 1/3, 2/4, 1/5, 2/6, \ldots.$
   3) $1, 3, 6, 10, \ldots$
2. If $S_n = 1 + 1/2 + 1/3 + \ldots + 1/(2^n - 1)$ prove

   $n/2 < S_n < n$ for $n >= 2$.

Strong induction

- For our coins problem, needed not just P(k-1), but P(k-3), and to look at three cases.
- Mathematical Induction principle:
  - $(P(0) \wedge \forall k \in \mathbb{N}\ \ P(k) \rightarrow P(k+1)) \ \rightarrow \forall x \in \mathbb{N}\ P(x)$
- Strong Induction principle:
  - $\left(\exists b \in \mathbb{N}\ \ \forall c \in \mathbb{N}\ (0 \leq c \wedge c \leq b \rightarrow \ P(c))\right)$
    $\wedge \forall k > b\ \ (\forall i\ \in \{0, \ldots, k-1\}\ P(i)) \ \rightarrow\ P(k))$
    $\rightarrow \forall x \in \mathbb{N}\ P(x)$
- Strong induction seems stronger…
  - But in fact, mathematical induction, strong induction and well-order principles are equivalent to each other.
  - So choose the most convenient one.

Puzzle: coins

- A not-too-far-away country recently got rid of a penny coin, and now everything needs to be rounded to the nearest multiple of 5 cents…
  - Suppose that instead of just dropping the penny, they would introduce a 3 cent coin.
    - Like British three pence.
  - What is the largest amount that cannot be paid by using only existing coins (5, 10, 25) and a 3c coin?

7c
Any number n >7 can be paid with 3,5,10,25 coins (even just 3 and 5).

Strong induction

- Strong Induction principle (general form):
  - $(\exists b \in \mathbb{N}\ \forall c \in \mathbb{N}\ (a \leq c \wedge c \leq b \rightarrow \mathrm{P(c)})$
    $\wedge \forall\, k > b\ \ (\forall\, i\ \in \{a, \dots, k-1\}\ \mathrm{P(i)}) \rightarrow \mathrm{P(k)})$
    $\rightarrow \forall x \in \mathbb{N}\ (x \geq a \rightarrow\ P(x))$
- Coins: $\forall x \in \mathbb{N}$, if x >7 then $\exists\ y, z \in \mathbb{N}$ such that x = 3y+5z.
  - P(n): $\exists\ y, z \in \mathbb{N}\ \ n = 3y + 5z$. Also, a=8.
  - Base cases: b = 10, so $c \in \{8,9,10\}$
    - n=8. $8 = 3 \cdot 1 + 5 \cdot 1$, so y=1, z=1.
    - n=9. 9=3$\cdot$ 3, y=3, z=0
    - n=10. 10=5 $\cdot$ 5. y=0, z=2.
  - Induction hypothesis: Let k be an arbitrary integer such that $k > 10$. Assume that for all $i \in \mathbb{N}$ such that $8 \leq i\ < k\ \exists\ y_i, z_i \in \mathbb{N}\ \ \ i = 3y_i + 5z_i$
  - Induction step. Show that induction hypothesis implies that $\exists\ y, z \in \mathbb{N}\ \ k = 3y + 5z$
    - Since $k \geq b,\ \ k - 3 \geq a$. So by induction hypothesis $\exists\ y_{k-3}, z_{k-3} \in \mathbb{N}\ \ \ k - 3 = 3y_{k-3} + 5z_{k-3}$. Now take z=$z_{k-3}$ and y = y_{k-3} +1. Then k = 3y+5z.
  - By strong induction, get that for all x > 7, $\exists\ y, z \in \mathbb{N}$ such that x = 3y+5z.

Puzzle: all horses are white

- Claim: all horses are white.
- Proof (by induction):
  - P(n): any n horses are white.
  - Base case: P(0) holds vacuously
  - Induction hypothesis: any k horses are white.
  - Induction step: if any k horses are white, then any k+1 horses are white.
    - Take an arbitrary set of k+1 horses. Take a horse out.
      - The remaining k horses are white by induction hypothesis.
    - Now put that horse back in, and take out another horse.
      - Remaining k horses are again white by induction hypothesis.
    - Therefore, all the k+1 horses in that set are white.
  - By induction, all horses are white.