**Cartesian products**

- **Cartesian product** of $A$ and $B$ is a set of all pairs of elements with the first from $A$, and the second from $B$:
  - $A \times B = \{(x, y) | x \in A, \ y \in B\}$
  - $A = \{1,2,3\}$, $B = \{a,b\}$
  - $A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$
  - $A = \{1,2\}$, $A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$

- Order of pairs does not matter, order within pairs does: $A \times B \neq B \times A$.

- Number of elements in $A \times B$ is $|A \times B| = |A| \cdot |B|$

- Can define the Cartesian product for any number of sets:
  - $A_1 \times A_2 \times \cdots \times A_k = \{(x_1, x_2, \ldots, x_k) | x_1 \in A_1 \ldots x_k \in A_k\}$
  - $A = \{1,2,3\}$, $B = \{a,b\}$, $C = \{3,4\}$
  - $A \times B \times C = \{(1, a, 3), (1, a, 4), (1, b, 3), (1, b, 4), (2, a, 3), (2, a, 4), (2, b, 3), (2, b, 4), (3, a, 3), (3, a, 4), (3, b, 3), (3, b, 4)\}$
Relations

- **A relation** is a subset of a Cartesian product of sets.
  - If of two sets (set of pairs), call it a **binary** relation.
  - Of 3 sets (set of triples), **ternary**. Of k sets (set of tuples), **k-ary**

- A={1,2,3}, B={a,b}
  - \( A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}\)
  - \( R = \{(1, a), (2, b), (3, a), (3, b)\} \) is a relation. So is \( R=\{(1,b)\} \).

- A={1,2},
  - \( A \times A = \{(1,1), (1,2), (2,1), (2,2)\}\)
  - \( R=\{(1,1), (2,2)\} \) (all pairs (x,y) where x=y)
  - \( R=\{(1,1),(1,2),(2,2)\} \) (all pairs (x,y) where \( x \leq y \)).

- A=PEOPLE
  - COUPLES =\{(x,y) \mid Loves(x,y)\}\)
  - PARENTS =\{(x,y) \mid Parent(x,y)\}\)

- A=PEOPLE, B=DOGS, C=PLACES
  - WALKS = \{(x,y,z) \mid x \text{ walks } y \text{ in } z\}
    - Jane walks Buddy in Bannerman park.
Databases and predicates

- In a database, store relations as tables.
- Then ask queries as predicate logic formulas
  - Return the set of all database elements satisfying the formula.
  - “Return first names of all profs who teach MWF “
  - $Q(fn)$:
    $\exists \ln \exists o \ ProfData(fn, ln, o)$
    $\land \exists c, t, r CourseData(c, "MWF", t, r, ln)$
Functions

- **A function** \( f : X \to Y \) is a relation on \( X \times Y \) such that for every \( x \in X \) there is at most one \( y \in Y \) for which \( (x, y) \) is in the relation.
  - Usual notation: \( f(x) = y \)
    - \( y \) is an **image** of \( x \) under \( f \).
  - \( X \) is the **domain** of \( f \)
  - \( Y \) is the **codomain** of \( f \)
  - **Range** of \( f \) (image of \( X \) under \( f \)):
    - \( \{ y \in Y \mid \exists x \in X, f(x) = y \} \)
  - **Preimage** of a given \( y \in Y \):
    - \( \{ x \in X \mid f(x) = y \} \)
    - Preimage of \( b \) is \( \{2,3\} \).

This \( R \) is not a function

This \( R \) is a function with domain \( \{1,2,3,4\} \), codomain \( \{a,b,c\} \) and range \( \{a,b\} \)
Functions

- A function $f: X \to Y$ is
  - **Total**: $\forall x \in X \exists y \in Y f(x) = y$
    - $f: \mathbb{Z} \to \mathbb{Z}$
    - $f(x) = x + 1$ is total.
    - $f(x) = \frac{100}{x}$ is not total.
  
  - **Onto**: $\forall y \in Y \exists x \in X f(x) = y$
    - $f(x) = x + 1$ is onto over $\mathbb{Z}$, but not over $\mathbb{N}$
    - $f(x) = 5x$ is not onto ($\mathbb{Z}$)

- **One-to-one**: $\forall x_1, x_2 \in X f(x_1) = f(x_2) \to x_1 = x_2$
  - $f(x) = x + 1$ is one-to-one.
  - $f(x) = x^2$ is not one-to-one

- **Bijection**: both one-to-one and onto.
  - $f(x) = x + 1$ is a bijection over $\mathbb{Z}$. 
Functions

• **An inverse** of $f$ is $f^{-1}: Y \rightarrow X$, such that $f^{-1}(y) = x$ iff $f(x) = y$
  
  - $f(x) = x + 1$, $f^{-1}(y) = y - 1$
  - Only one-to-one functions have an inverse

• **Composition** of $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ is $g \circ f: X \rightarrow Z$ such that $(g \circ f)(x) = g(f(x))$
  
  - $f(x) = \frac{x}{5}$, $g(x) = \lceil x \rceil$, over $\mathbb{R}$
    - $\lceil x \rceil$ is ceiling: $x$ rounded up to nearest integer.

  - $(g \circ f)(x) = g(f(x)) = \left\lceil \frac{x}{5} \right\rceil$
  
  - $(f \circ g)(x) = f(g(x)) = \left\lfloor \frac{x}{5} \right\rfloor$

  - $(g \circ f)(12.5) = \left\lfloor 2.5 \right\rfloor = 3$. $(f \circ g)(12.5) = 13/5 = 2.6$
    - Order matters!
Puzzle: the barber club

• In a certain barber’s club,
  – Every member has shaved at least one other member
  – No member shaved himself
  – No member has been shaved by more than one member
  – There is a member who has never been shaved.

• **Question**: how many barbers are in this club?