COMP 1002

Logic for Computer Scientists

Lecture 18
Cartesian products

- **Cartesian product** of A and B is a set of all pairs of elements with the first from A, and the second from B:
  - \( A \times B = \{(x, y) | x \in A, \ y \in B\} \)
  - \( A = \{1,2,3\}, \ B = \{a,b\} \)
  - \( A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\} \)
  - \( A = \{1,2\}, \ A \times A = \{(1,1), (1,2), (2,1), (2,2)\} \)

- Order of pairs does not matter, order within pairs does: \( A \times B \neq B \times A \).

- Number of elements in \( A \times B \) is \(|A \times B| = |A| \cdot |B|\)

- Can define the Cartesian product for any number of sets:
  - \( A_1 \times A_2 \times \cdots \times A_k = \{(x_1, x_2, \ldots, x_k) | x_1 \in A_1, \ldots, x_k \in A_k\} \)
  - \( A = \{1,2,3\}, \ B = \{a,b\}, \ C = \{3,4\} \)
  - \( A \times B \times C = \{(1, a, 3), (1, a, 4), (1, b, 3), (1, b, 4), (2, a, 3), (2, a, 4), (2, b, 3), (2, b, 4), (3, a, 3), (3, a, 4), (3, b, 3), (3, b, 4)\} \)

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<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
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<tbody>
<tr>
<td>1</td>
<td>(1,a)</td>
<td>(1,b)</td>
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<tr>
<td>2</td>
<td>(2,a)</td>
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<tr>
<td>3</td>
<td>(3,a)</td>
<td>(3,b)</td>
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Relations

- **A relation** is a subset of a Cartesian product of sets.
  - If of two sets (set of pairs), call it a **binary** relation.
  - Of 3 sets (set of triples), **ternary**. Of k sets (set of tuples), **k-ary**

- \( A=\{1,2,3\}, \; B=\{a,b\} \)
  - \( A \times B = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\} \)
  - \( R = \{(1,a),(2,b),(3,a),(3,b)\} \) is a relation. So is \( R = \{(1,b)\} \).

- \( A=\{1,2\}, \)
  - \( A \times A = \{(1,1),(1,2),(2,1),(2,2)\} \)
  - \( R = \{(1,1),(2,2)\} \) (all pairs (x,y) where x=y)
  - \( R = \{(1,1),(1,2),(2,2)\} \) (all pairs (x,y) where \( x \leq y \)).

- \( A=\text{PEOPLE} \)
  - \( \text{COUPLES} = \{(x,y) \mid \text{Loves}(x,y)\} \)
  - \( \text{PARENTS} = \{(x,y) \mid \text{Parent}(x,y)\} \)

- \( A=\text{PEOPLE}, \; B=\text{DOGS}, \; C=\text{PLACES} \)
  - \( \text{WALKS} = \{(x,y,z) \mid x \text{ walks } y \text{ in } z\} \)
    - Jane walks Buddy in Bannerman park.
Databases and predicates

In a database, store relations as tables.

Then ask queries as predicate logic formulas

- Return the set of all database elements satisfying the formula.
- "Return first names of all profs who teach MWF" 
  - $Q(fn)$:
    $\exists ln \exists o\ ProfData(fn, ln, o) \land \exists c, t, r\ CourseData(c, "MWF", t, r, ln)$
Functions

• A function \( f : X \to Y \) is a relation on \( X \times Y \) such that for every \( x \in X \) there is at most one \( y \in Y \) for which \( (x, y) \) is in the relation.
  – Usual notation: \( f(x) = y \)
    • \( y \) is an image of \( x \) under \( f \).
  – \( X \) is the domain of \( f \)
  – \( Y \) is the codomain of \( f \)
  – Range of \( f \) (image of \( X \) under \( f \)):
    • \( \{ y \in Y \mid \exists x \in X, f(x) = y \} \)
  – Preimage of a given \( y \in Y \):
    • \( \{ x \in X \mid f(x) = y \} \)
      – Preimage of \( b \) is \( \{2,3\} \).

This \( R \) is not a function

This \( R \) is a function with domain \( \{1,2,3,4\} \), codomain \( \{a,b,c\} \) and range \( \{a,b\} \).
Functions

- **A function** $f: X \rightarrow Y$ is
  - **Total**: $\forall x \in X \exists y \in Y \ f(x) = y$
    - $f: \mathbb{Z} \rightarrow \mathbb{Z}$
    - $f(x) = x + 1$ is total.
    - $f(x) = \frac{100}{x}$ is not total.
  
- **Onto**: $\forall y \in Y \exists x \in X \ f(x) = y$
  - $f(x) = x + 1$ is onto over $\mathbb{Z}$, but not over $\mathbb{N}$
  - $f(x) = 5x$ is not onto ($\mathbb{Z}$)
  
- **One-to-one**: $\forall x_1, x_2 \in X \ f(x_1) = f(x_2) \rightarrow x_1 = x_2$
  - $f(x) = x + 1$ is one-to-one.
  - $f(x) = x^2$ is not one-to-one

- **Bijection**: both one-to-one and onto.
  - $f(x) = x + 1$ is a bijection over $\mathbb{Z}$. 
Functions

• An **inverse** of $f$ is $f^{-1}: Y \rightarrow X$, such that $f^{-1}(y) = x$ if $f(x) = y$
  
  - $f(x) = x + 1$, $f^{-1}(y) = y - 1$
  - Only one-to-one functions have an inverse

• **Composition** of $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ is $g \circ f: X \rightarrow Z$ such that $(g \circ f)(x) = g(f(x))$
  
  - $f(x) = \frac{x}{5}$, $g(x) = [x]$, over $\mathbb{R}$
    - $[x]$ is ceiling: $x$ rounded up to nearest integer.
      
  - $(g \circ f)(x) = g(f(x)) = \left\lceil \frac{x}{5} \right\rceil$
  - $(f \circ g)(x) = f(g(x)) = \frac{[x]}{5}$
  - $(g \circ f)(12.5) = [2.5] = 3$. $(f \circ g)(12.5) = 13/5 = 2.6$
    - Order matters!
Puzzle: the barber club

• In a certain barber’s club,
  – Every member has shaved at least one other member
  – No member shaved himself
  – No member has been shaved by more than one member
  – There is a member who has never been shaved.

• **Question:** how many barbers are in this club?