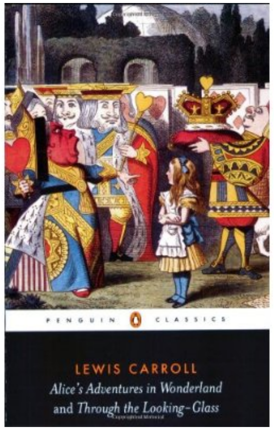
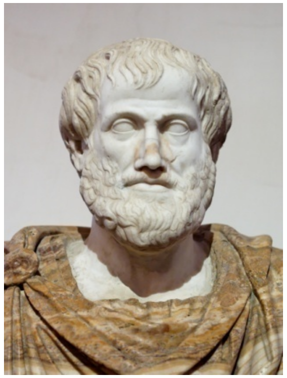


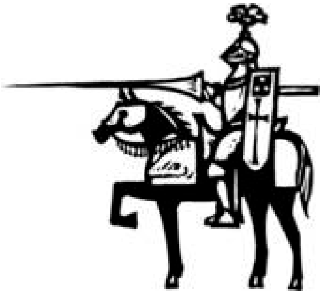


COMP 1002

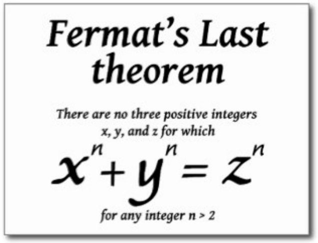
Intro to Logic for Computer Scientists

Lecture 13

B 5 2 J



Puzzle 11

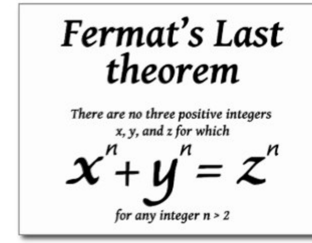


- Let
  $S = \{x \in \mathbb{N} \mid x \textit{ is even}\} \cap \{x \in \mathbb{N} \mid x \textit{ is odd}\}$

- Prove or disprove:

$$\forall x \in S, \qquad x \textit{ does not divide } x^2$$

Puzzle 11

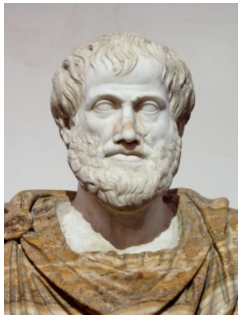


- Let $S = \{x \in \mathbb{N} \mid x\ is\ even\} \cap \{x \in \mathbb{N} | x\ is\ odd\}$
  - S = Ø
- Prove or disprove:

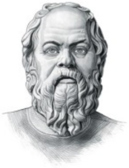
$$\forall x \in S, \qquad x\ does\ not\ divide\ x^2$$

  - Let P(x)= "$x\ does\ not\ divide\ x^2$"
  - To disprove, can give a counterexample
    - Find an element in S such that P(x) is true…
    - But there is no such element in S, because there are no elements in S at all!
  - To prove, enough to check that it holds for all elements of S.
    - There is none, so it does hold for every element in S.
  - Another way: Since S is defined as a subset of natural numbers, can read $\forall x \in S\ P(x)$ as $\forall x \in \mathbb{N}\ \big(x \in S \to P(x)\big)$.
    - Since "$x \in S$" is always false, $x \in S \to P(x)$ is true for every $x \in \mathbb{N}$
  - Call a statement $\forall x \in \emptyset\ P(x)$ **vacuously true.**

Universal Modus Ponens

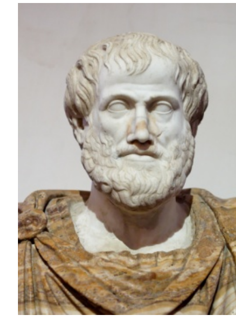


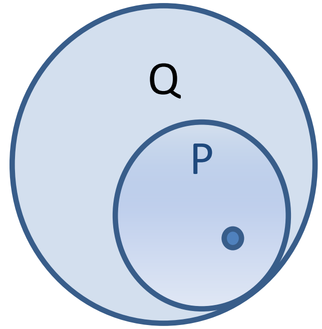
- All men are mortal
- Socrates is a man
- Therefore, Socrates is mortal



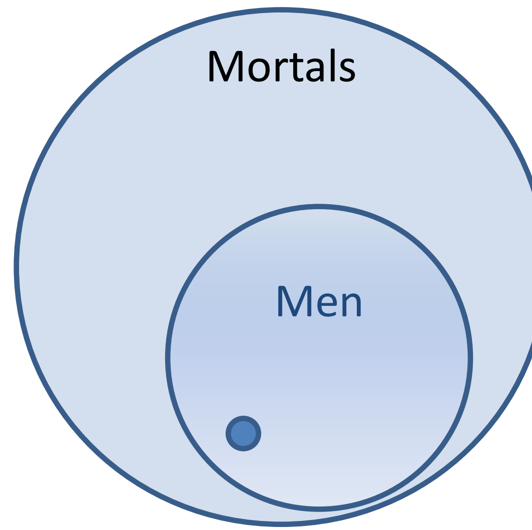
- All cats like fish
- Molly likes fish
- Therefore, Molly is a cat

Universal Modus Ponens





- $\forall x,\ P(x)\ \rightarrow Q(x)$
- $P(a)$
- -------------------------
- $Q(a)$

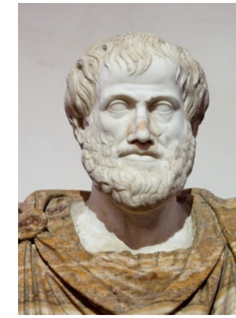


- All men are mortal ($\forall x,\ Man(x) \rightarrow Mortal(x)$)
- Socrates is a man $(Man(Socrates))$
- Therefore, Socrates is mortal $(Mortal(Socrates))$

- All numbers are either odd or even
- 2 is a number
- Therefore, 2 is either odd or even.

- All trees drop leaves
- Pine does not drop leaves
- Therefore, pine is not a tree

Universal Modus Ponens



- All men are mortal
- Socrates is a man
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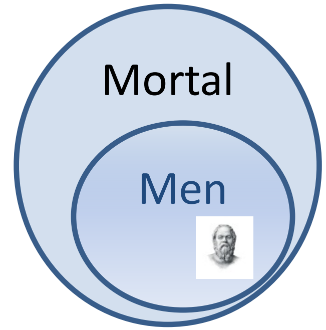


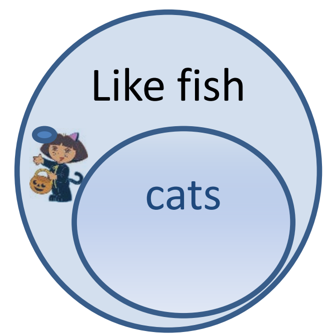
- All cats like fish
- Molly likes fish
- Therefore, Molly is a cat

$(\forall x)\ Cat(x) \rightarrow like\_fish(x)$

- $like\_fish(Molly)$

____

Cat(Molly) X



Instantiation/generalization

- In general, if $\forall\ x \in S\ \ F(x)$ is true for some formula $F(x)$, if you take any specific element $a \in S$, then $F(a)$ must be true.
  - This is called the **universal instantiation** rule.
    - $\forall x \in \mathbb{N}\ \ (x > -1)$
    - $\therefore\ \ 5 > -1$
- If you prove $F(a)$ without any assumptions about a other than $a \in S$, then $\forall x \in S, F(x)$
  - This is called **universal generalization**.

Instantiation/generalization

- If you can find an element $a \in S$ such that $F(a)$, then $\exists x \in S, F(x)$
  - This is called **existential generalization**.
- Alternatively, if $\exists\ x \in S\ \ F(x)$ is true, then you can give that element of S for which $F(x)$ is true a name, as long as that name has not been used elsewhere.
  - This is called the **existential instantiation** rule.
    - $\exists x \in \mathbb{N}\ \ (x - 5 = 0)$
    - $\therefore\ \ k = 0 + 5$

Existential instantiation

- If $\exists\, x \in S \;\; F(x)$ is true, then you can give that element of S for which $F(x)$ is true a name, as long as that name has not been used elsewhere.
  - "Let n be an even number. Then n=2k for some k".
    - $\forall x \in \mathbb{N} \;\; Even(x) \rightarrow \;\; \exists\, y \in \mathbb{N} \;\; (x = 2 * y)$
  - Important to have a new name!
    - "Let n and m be two even numbers. Then n=2k and m=2k" is wrong!
    - $\forall x_1, x_2 \;\in \mathbb{N} \;\; Even(x_1) \wedge Even(x_2) \rightarrow$ $\exists\, y_{1,} y_2 \in \mathbb{N} \;\; (x_1 = 2 * y_1) \wedge (x_2 = 2 * y_2)$
    - "Let n and m be two even numbers. Then n=2k and m=2ℓ"

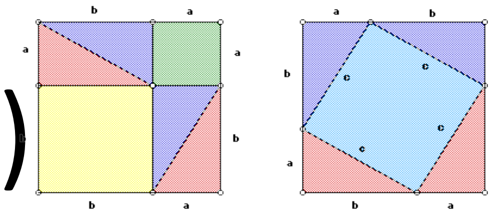
Other inference rules

- Combining universal instantiation with tautologies, get other types of arguments:

$p \rightarrow q$	• $\forall x\, P(x) \rightarrow Q(x)$	For any x, if $x > 3$, then $x > 2$
$q \rightarrow r$	• $\forall x\, Q(x) \rightarrow R(x)$	For any x, if $x > 2$, then $x \neq 1$
______	________________	________________
$\therefore p \rightarrow r$	$\therefore \forall x\, P(x) \rightarrow R(x)$	$\therefore$ For any x, if $x > 3$, then $x \neq 1$

- (This particular rule is called “transitivity”)

Types of proofs (1)



- Direct proof of $\forall x\ F(x)$
  - Show that $F(x)$ holds for arbitrary x, then use universal generalization.
    - Often, $F(x)$ is of the form $G(x) \rightarrow H(x)$
  - Example: A sum of two even numbers is even.

- Proof by cases
  - If can write $\forall x\ F(x)$ as $\forall x(G_1(x) \vee G_2(x) \vee \cdots \vee G_k(x)) \rightarrow H(x)$, prove $(G_1(x) \rightarrow H(x)) \wedge (G_2(x) \rightarrow H(x)) \wedge \cdots \wedge (G_k(x) \rightarrow H(x))$

  Example: $x \in$ {days in August}

  (∀*x)(*rain(x) V sunny(x) V foggy(x)) →hot(x)

  you may prove

  (∀x)

  (rain(x)→hot(x))

  ∧ (sunny(x) →hot(x))

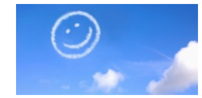
  ∧ (foggy(x)→hot(x)))

Type of Proof (2)

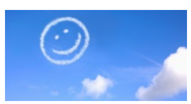
- Proof by contraposition
  - To prove $\forall x \;\; G(x) \rightarrow H(x)$, prove $\forall x \neg H(x) \rightarrow \neg G(x)$
  - Example: (∀x)*(even(x) →integer(x)), prove*
    (∀x)(¬integer(x) → ¬even(x))

- Proof by contradiction
  - To prove $\forall x \, F(x)$, prove $\forall x \neg F(x) \rightarrow FALSE$
  - Example: $\sqrt{2}$ is not a rational number.
  - Example: There are infinitely many primes.

Puzzle: better than nothing

- Nothing is better than eternal bliss 
- A burger is better than nothing 

- Therefore, a burger is better than eternal bliss.

 $\leq$ $\leq$ 

Is there anything wrong with this argument?

The premise: "Nothing is better than eternal bliss" is not true.