COMP 1002

Intro to Logic for Computer Scientists

Lecture 13
Puzzle 11

• Let
  \[ S = \{ x \in \mathbb{N} \mid x \text{ is even} \} \cap \{ x \in \mathbb{N} \mid x \text{ is odd} \} \]

• Prove or disprove:
  \[ \forall x \in S, \quad x \text{ does not divide } x^2 \]
Puzzle 11

• Let $S = \{x \in \mathbb{N} | x \text{ is even}\} \cap \{x \in \mathbb{N} | x \text{ is odd}\}$
  – $S = \emptyset$

• Prove or disprove:
  \[
  \forall x \in S, \quad x \text{ does not divide } x^2
  \]
  – Let $P(x) = \text{“}x \text{ does not divide } x^2\text{“}$
  – To disprove, can give a counterexample
    • Find an element in $S$ such that $P(x)$ is true...
    • But there is no such element in $S$, because there are no elements in $S$ at all!
  – To prove, enough to check that it holds for all elements of $S$.
    • There is none, so it does hold for every element in $S$.
  – Another way: Since $S$ is defined as a subset of natural numbers, can read
    $\forall x \in S P(x)$ as $\forall x \in \mathbb{N} \left( x \in S \rightarrow P(x) \right)$.
    • Since "$x \in S$" is always false, $x \in S \rightarrow P(x)$ is true for every $x \in \mathbb{N}$
  – Call a statement $\forall x \in \emptyset P(x)$ vacuously true.
Universal Modus Ponens

• All men are mortal
• Socrates is a man
• Therefore, Socrates is mortal

• All cats like fish
• Molly likes fish
• Therefore, Molly is a cat
Universal Modus Ponens

- $\forall x, P(x) \rightarrow Q(x)$
- $P(a)$
- -------------------------
- $Q(a)$

- All men are mortal ($\forall x, \text{Man}(x) \rightarrow \text{Mortal}(x)$)
- Socrates is a man ($\text{Man}(\text{Socrates})$)
- Therefore, Socrates is mortal ($\text{Mortal}(\text{Socrates})$)

- All numbers are either odd or even
- 2 is a number
- Therefore, 2 is either odd or even.

- All trees drop leaves
- Pine does not drop leaves
- Therefore, pine is not a tree
Universal Modus Ponens

- All men are mortal
- Socrates is a man
- Therefore, Socrates is mortal

- All cats like fish
- Molly likes fish
- Therefore, Molly is a cat

\[(\forall x) \text{Cat}(x) \rightarrow \text{like}_\text{fish}(x)\]

\[\text{like}_\text{fish}(\text{Molly})\]

\[\text{Cat}({\text{Molly}}) \quad X\]
Instantiation/generalization

• In general, if $\forall x \in S \ F(x)$ is true for some formula $F(x)$, if you take any specific element $a \in S$, then $F(a)$ must be true.
  – This is called the **universal instantiation** rule.
    • $\forall x \in \mathbb{N} \ (x > -1)$
    • $\therefore \ 5 > -1$

• If you prove $F(a)$ without any assumptions about $a$ other than $a \in S$, then $\forall x \in S, F(x)$
  – This is called **universal generalization**.
Instantiation/generalization

• If you can find an element \( a \in S \) such that \( F(a) \), then \( \exists x \in S \), \( F(x) \)
  – This is called existential generalization.

• Alternatively, if \( \exists x \in S \) \( F(x) \) is true, then you can give that element of \( S \) for which \( F(x) \) is true a name, as long as that name has not been used elsewhere.
  – This is called the existential instantiation rule.

  • \( \exists x \in \mathbb{N} \) (\( x - 5 = 0 \))
  • \( \therefore k = 0 + 5 \)
Existential instantiation

- If $\exists x \in S \ F(x)$ is true, then you can give that element of $S$ for which $F(x)$ is true a name, as long as that name has not been used elsewhere.

  - “Let $n$ be an even number. Then $n=2k$ for some $k$”.
    - $\forall x \in \mathbb{N} \ Even(x) \to \exists y \in \mathbb{N} \ (x = 2 \times y)$
    - Important to have a new name!
    - “Let $n$ and $m$ be two even numbers. Then $n=2k$ and $m=2k$” is wrong!

- $\forall x_1, x_2 \in \mathbb{N} \ Even(x_1) \land Even(x_2) \to \exists y_1, y_2 \in \mathbb{N} \ (x_1 = 2 \times y_1) \land (x_2 = 2 \times y_2)$
- “Let $n$ and $m$ be two even numbers. Then $n=2k$ and $m=2\ell$”
Other inference rules

• Combining universal instantiation with tautologies, get other types of arguments:

\[
p \rightarrow q \quad \forall x \ P(x) \rightarrow Q(x) \quad \text{For any } x, \text{ if } x > 3, \text{ then } x > 2
\]

\[
q \rightarrow r \quad \forall x \ Q(x) \rightarrow R(x) \quad \text{For any } x, \text{ if } x > 2, \text{ then } x \neq 1
\]

\[
\therefore \ p \rightarrow r \quad \therefore \ \forall x \ P(x) \rightarrow R(x) \quad \therefore \ \text{For any } x, \text{ if } x > 3, \text{ then } x \neq 1
\]

• (This particular rule is called “transitivity”)
Types of proofs (1)

- **Direct proof of** $\forall x F(x)$
  - Show that $F(x)$ holds for arbitrary $x$, then use universal generalization.
    - Often, $F(x)$ is of the form $G(x) \rightarrow H(x)$
    - Example: A sum of two even numbers is even.

- **Proof by cases**
  - If can write $\forall x F(x)$ as $\forall x(G_1(x) \lor G_2(x) \lor \cdots \lor G_k(x)) \rightarrow H(x)$, prove $(G_1(x) \rightarrow H(x)) \land (G_2(x) \rightarrow H(x)) \land \cdots \land (G_k(x) \rightarrow H(x))$

Example: $x \in \{\text{days in August}\}$

$$(\forall x)(\text{rain}(x) \lor \text{sunny}(x) \lor \text{foggy}(x)) \rightarrow \text{hot}(x)$$

you may prove

$$(\forall x)$$

$$(\text{rain}(x) \rightarrow \text{hot}(x))$$

$$(\text{sunny}(x) \rightarrow \text{hot}(x))$$

$$(\text{foggy}(x) \rightarrow \text{hot}(x))$$

)
Type of Proof (2)

• Proof by contraposition
  – To prove $\forall x \ G(x) \rightarrow H(x)$, prove $\forall x \ \neg H(x) \rightarrow \neg G(x)$
  – Example: $(\forall x)(\text{even}(x) \rightarrow \text{integer}(x))$, prove $(\forall x)(\neg \text{integer}(x) \rightarrow \neg \text{even}(x))$

• Proof by contradiction
  – To prove $\forall x \ F(x)$, prove $\forall x \ \neg F(x) \rightarrow FALSE$
  – Example: $\sqrt{2}$ is not a rational number.
  – Example: There are infinitely many primes.
Puzzle: better than nothing

• Nothing is better than eternal bliss
• A burger is better than nothing

Therefore, a burger is better than eternal bliss.

Is there anything wrong with this argument?

The premise: “Nothing is better than eternal bliss” is not true.