COMP 1002

Intro to Logic for Computer Scientists

Lecture 10
Puzzle 9

- Susan is 28 years old, single, outspoken, and very bright. She majored in philosophy. As a student she was deeply concerned with issues of discrimination and social justice and also participated in anti-nuke demonstrations.

*Please rank the following possibilities by how likely they are. List them from least likely to most likely. Susan is:*

1. a kindergarden teacher
2. works in a bookstore and takes yoga classes
3. an active feminist
4. a psychiatric social worker
5. a member of an outdoors club
6. a bank teller
7. an insurance salesperson
8. a bank teller and an active feminist
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Scenarios and sets

• Want to reason about more general scenarios
• Rather than just true/false, vary over objects:
  – even numbers, integers, primes
  – people that are and are not bank tellers,
  – pairs of animals in the same ecosystem...
• Want multiple properties of these objects:
  – an even number that is divisible by 4 and > 10,
  – a person that is also a bank teller...
Sets

• A **set** is a collection of objects.
  – \( S_1 = \{1, 2, 3\}, \ S_2 = \{\text{Cathy, Alaa, Keiko, Daniela}\} \)
  – \( S_3 = [-1, 2] \) (real numbers from -1 to 2, inclusive)
  – \( \text{PEOPLE} = \{x \mid x \text{ is a person living on Earth now}\} \)
    • \( \{x \mid \text{such that } x \ldots \} \) is called **set builder notation**
  – \( S_4 = \{(x,y) \mid x \text{ and } y \text{ are people, and } x \text{ is a parent of } y\} \)
  – \( \text{BANKTELLERS} = \{x \mid x \text{ is a person who is a bank teller}\} \)

• The order of elements does not matter.
• There are no duplicates.
Special sets

- Notation for some special sets (much of which you are likely to have seen):
  - Empty set $\emptyset$
  - Natural numbers $\mathbb{N} = \{1, 2, 3, \ldots \}$ (sometimes with 0)
  - Integers $\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots \}$
  - Rational numbers $\mathbb{Q} = \left\{ \frac{m}{n} \mid m, n \text{ in } \mathbb{Z}, n \neq 0 \right\}$
  - Real numbers $\mathbb{R}$
  - Complex numbers $\mathbb{C}$
Set elements

• \( a \in S \) means that an element \( a \) is in a set \( S \), and \( a \notin S \) that \( a \) is not in \( S \). That is, \( a \in S \equiv \neg (a \notin S) \)
  – Susan \( \in \) PEOPLE. Susan \( \notin \) BANKTELLERS
  – 0.23 \( \in \) \([-1, 2]\). 3.54 \( \notin \) \([-1,2]\)

• Also, write \( x \in S \) for a variable \( x \).
  – BANKTELLERS = \{ \ x \in PEOPLE \ | \ x \text{ is a bank teller} \}\n
• How do we generalize sentences like “\( x \) is a bank teller”, where \( x \) is an element of some set?
Predicates

- A **predicate** \( P(x_1, ..., x_n) \) is a “proposition with variables”, where values of the variables \( x_1, ..., x_n \) come from some sets \( S_1, ..., S_n \), called their **domains** or **universes**.
  - \( P(x) \) is true for some values of \( x \in S \), and false for the rest.
    - Even(x) for \( x \in \mathbb{Z} \), Feminist(y) for \( y \in PEOPLE \).
    - Here, domain of \( x \) is \( \mathbb{Z} \), and domain of \( y \) is \( PEOPLE \).
    - Even(y) is not defined for \( y \in PEOPLE \), only for elements of \( \mathbb{Z} \).
  - A predicate can have several variables:
    - \( x > y \), for \( x, y \in \mathbb{R} \)
    - Divides(x, y), which is true for \( x, y \in \mathbb{Z} \) such that \( x \) divides \( y \).

- When all variables in a predicate are replaced with specific elements (**instantiated**), the predicate becomes a proposition.
  - “Even(3)” is false. “Feminist(Susan)” is true.
Predicates

• We can make formulas out of predicates the same way as we did for propositions, but now our formulas have free variables:
  – \( \text{Even}(x) \lor \text{Divides}(3, x) \rightarrow \neg \text{Prime}(x) \)
  – \( \text{Feminist}(x) \land \text{Bankteller}(x) \)
  – Now scenarios can correspond to values of \( x \).
    • The first formula is false for \( x=2 \), because \( \text{Even}(2) = \text{true} \), but \( \neg \text{Prime}(2) = \text{false} \).

• This is called **predicate logic** (or **first-order logic**), as opposed to propositional logic we did so far.
Quantifiers: universal (∀)

• Theorems often look like this: “For all x, the following is true”, and then a formula with x as a free variable.
  – For all $x \in \mathbb{Z}$, $\text{Divides}(6, x) \rightarrow \text{Divides}(3, x)$
  – For all $n \in \mathbb{N}$, $n > 4, \ 2^n > n^2$

• We write this in predicate logic using a universal quantifier (written as ∀):
  – $\forall x \in \mathbb{Z}, \ \text{Divides}(6, x) \rightarrow \text{Divides}(3, x)$
  – $\forall n \in \mathbb{N}, \ n > 4 \rightarrow 2^n > n^2$
• In general, for every formula $F$ of predicate logic with a free variable $x$, we can write
\[ \forall x \in S, \; F(x) \]

– The formula “$\forall x \in S, \; F(x)$” is true if and only if $F(a)$ is true for every $a \in S$.

– That is, if $a_1, a_2, \ldots, a_n, \ldots$ is a list of all elements of $S$, then $\forall x \in S, \; F(x)$ is true if and only if $F(a_1) \land F(a_2) \land \cdots \land F(a_n) \land \cdots$ is true.

– If there are no more free variables or quantifiers in $F$, then $\forall x \in S, \; F(x)$ is true if and only if $F(a_1) \land F(a_2) \land \cdots \land F(a_n) \land \cdots$ is a tautology.
Negating the universal

• What is the negation of “All”? When would a statement “∀𝑥 ∈ 𝑆, 𝐹(𝑥)” be false?
  – All girls hate math.
  – No!
    • All girls love math?
    • Some girls do not hate math!

  – Everybody in O’Brian family is tall
    • No, Jenny is O’Brian and she is quite short.

  – It is foggy all the time, every day in St. John’s
    • No, sometimes it is not foggy (like today).
Quantifiers: existential (∃)

• To prove that something is not always true, we give a counterexample. In predicate logic, use existential quantifier \( \exists \).

• \( \exists x \in S, F(x) \) is true if and only if there exist some \( a \in S \) such that \( F(a) \) is true (and we don’t care for the rest). That is, when \( F(a_1) \lor F(a_2) \lor \ldots \lor F(a_n) \lor \ldots \) is true.
  – \( \exists t \in \text{TIMESLOTS}, \text{Scheduled} (\text{COMP1002}, t) \land \text{Scheduled} (\text{COMP1000}, t) \)
  – \( \exists x \in \mathbb{N}, \text{Even}(x) \land \text{Prime}(x) \).

• \( \neg \forall x \in S, F(x) \equiv \exists x \in S, \neg F(x) \)

• Once a variable is quantified, it is no longer free.
  – \( x \) is free in \( \text{Even}(x) \land \text{Prime}(x) \),
  – But \( \exists x \in \mathbb{N}, \text{Even}(x) \land \text{Prime}(x) \) has no free variables.
Quantifiers in English

• Universal quantifier: usually “every”, “all”, “each”, “any”.
  – Every day it is foggy. Each number is divisible by 1.

• Existential quantifier: “some”, “a”, “exists”
  – Some students got 100% on both labs.
  – There exists a prime number greater than 100.

• The word “any” can mean either!
Quantifiers in English: “any”

• “Any” can mean “every”:
  – Any student in our class knows logic 😊
  – Every student in our class knows logic. 😊

• But “any” can also mean “some”!
  – I will be happy if I do well on every quiz. 😊
  – I will be happy if I do well on any quiz. 😊😊
Puzzle 10

• The first formulation of the famous liar’s paradox, attributed to a Cretan philosopher Epimenides, stated

“All Cretans are liars”.

Is this really a paradox?