A Change for the Better? Assessing the Computational Cost of Re-representation

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Abstract

The ability to re-represent information—i.e., to see things in new ways—is essential for human reasoning, creativity, and learning. It forms the foundation of insight problem solving and scientific explanation, and is hypothesized to play a pivotal role in concept development in children. Re-representation is useful because it allows a cognizer to make sense of things in ways that were previously impossible. Yet, invoking this operation can quickly become computationally intractable in light of the combinatorial explosion of re-representation options to consider. Although this intractability may explain why discovering useful ways of re-representing information can be cognitively challenging at times (as in insight puzzles and creativity), it seems difficult to reconcile with automatic and apparently effortless forms of re-representation (as in everyday analogizing and children's development of concepts). To get more insight into the conditions that can make re-representation tractable, we performed computational complexity analyses of a formal model of re-representation as invoked in analogy derivation. We will discuss how our complexity results can help explain when and why re-representation can be invoked effectively and efficiently.

Keywords: Re-representation; Analogy; Structure Mapping; Computational Complexity; Fixed-Parameter Tractability

Introduction

Many theories of cognitive abilities operate on mental representations of information, each of which assumes a particular encoding of relevant situations and concepts. As there are typically many possible encodings, one's initial representation may in fact be inappropriate for the task at hand, slowing down or even stopping the ability from functioning. In such cases, it is hypothesized that humans change encodings and re-represent stored information.

Re-representation has been invoked in many cognitive theories. It allows natural analogies that rely on semantic rather than syntactic matching (*e.g.*, "Bob running into the cave" is like "Alice walking into a room" ⇒ "Bob moves into the cave" is like "Alice moves into the room" (Gentner & Kurtz, 2006)). The powerful abstractions generated by such re-representation in turn have been hypothesized to underlie certain types of concept development in children (*e.g.*, the emergence of abstract relations and attributes: "X is hotter than Y" ⇒ "temperature(X) is greater than temperature(Y)" (Gentner, Rattermann, Markman, & Kotovosky, 1995)). More radical types of re-representation can in turn lead to totally new ways of envisioning particular situations and concepts, and thus have been invoked in theories of insight problem solving (Ohlsson, 1992), scientific discovery (Gentner et al., 1997), and creativity (Welling, 2007).

Investigating re-representation experimentally is difficult, but there is increasing evidence for its psychological reality (Gentner & Kurtz, 2006; Kurtz, 2006). This has motivated the development of computational theories of re-representation (*e.g.*, Ohlsson (1992); Yan, Forbus, and Gentner (2003); Krumnack, Gust, Kühnberger, and Schwering (2008)), which has raised the following conundrum: The combinatorial explosion of re-representation options that must be considered in such theories seems to be computational intractable. This intractability may explain why certain cognitive activities invoking re-representation like insight problem solving and creativity are challenging, but is at odds with how other forms of re-representation invoked in everyday reasoning or cognitive development seem so effortless and automatic.

In this paper, we assess the computational difficulty of a basic type of re-representation, namely individual predicate re-representation within Gentner's Structure Mapping Theory of analogy derivation (SMT) (Gentner, 1983; Yan et al., 2003). We give the first proof that such re-representation is computationally intractable, even when invoked in the context of incremental rather than general analogy derivation. This finding indicates that constraints on both the re-representation process and its inputs must be exploited to yield tractability. In the second part of this paper we illustrate a methodology suitable for identifying such constraints. We also discuss how our results can help explain when and why re-representation can be invoked effectively and efficiently.

Computational-level Models

Analogies are defined over concepts, which are represented in SMT by predicate-structures composed of entities (*e.g.*, SUN, PLANET) and predicates relating those entities (as well as other predicates) (*e.g.*, ATTRACTS(SUN, PLANET)). Predicate-structures are naturally represented as vertex-labelled directed acyclic graphs in which entities are leaves, predicates are internal vertices, and predicates are linked to their arguments by arcs (see part (a) of Figure 1).

An **analogy** "T is (like) a B", where B and T are predicate-structures, is a mapping from a portion of B to a portion of T that satisfies the following three conditions:

1. The mapping is **structurally consistent**, *i.e.*, matching relations must have matching arguments and any element in one predicate-structure matches at most one element in the other.

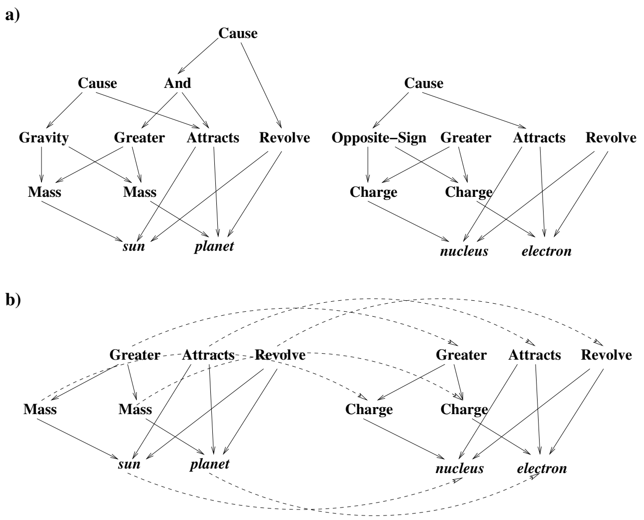

Figure 1: Analogy Derivation in Structure-Mapping Theory. (a) Two graph representations of predicate structures encoding descriptions of the solar system (left) and the Rutherford model of the atom (right). (b) An optimal analogy between the structures in (a), where dotted arrows indicate the mappings between corresponding pairs of predicates and objects.

2. **Relational focus**: The mapping must involve common predicates but need not involve common objects, *i.e.*, matched predicates must have the same type, argument, number and order but matched objects need not have the same name.
3. **Systematicity**: The mapping tends to match interconnected, deeply-nested predicate-substructures.

Let $val(A)$ be the systematicity of an analogy A. The most systematic analogy between a pair of predicate-structures is an **optimal analogy** (see part (b) of Figure 1).

Under SMT, re-representation of predicates is only invoked to better the analogical match between two given predicate-structures. As such, it relaxes identical-only predicate-type matches (*e.g.*, ATTRACTS $\rightarrow$ ATTRACTS) to allow selected non-identical matches (*e.g.*, WALK $\rightarrow$ MOVE). There are two classes of mechanisms for performing re-representations:

1. **Rule-guided** (part (a) of Figure 2): A predicate of type x can be re-represented as a predicate of type y if there is a rule $x \rightarrow y$. Collections of rules can be encoded as predicate-type similarity tables (represented explicitly (Holyoak & Thagard, 1989) or generated implicitly by predicate decomposition (Gentner et al., 1995)) or generalization lattices (in which the most specific predicate-types are at the bottom of the lattice and the most abstract predicate-types are at the top) (Winston, 1980).

2. **Context-guided** (part (b) of Figure 2): A predicate p of type x can be re-represented as a predicate of type y if p appears in a structural context immediately "outside" an existing analogy between two predicate-structures which, if p's type is changed, will allow an incremental addition to that analogy which increases its systematicity. The most basic type of context is a "hole", in which a pair of predicates in B and T have different types but both their arguments and parents have the same types and can be matched.

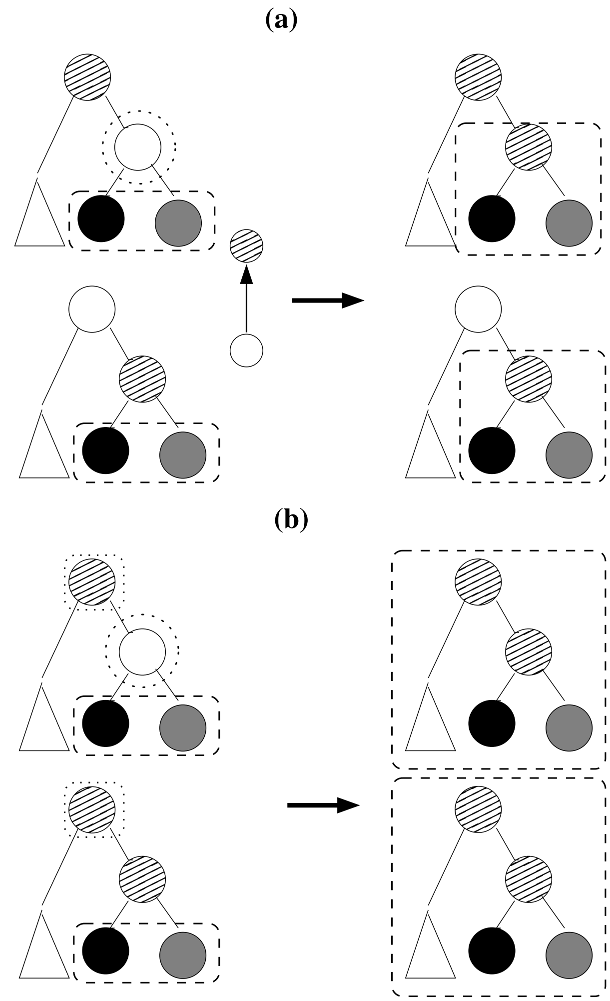

Figure 2: Re-Representation Mechanisms in Structure-Mapping Theory. (a) Rule-guided. (b) Context-guided. Analogically-matched regions are enclosed by dashed boxes.

Analogy derivation alternates with such re-representation until a satisfactory analogy is reached. In any one round of re-representations, it is assumed that each predicate in the given predicate-structures can change at most once. Though we focus here on single-predicate re-representation, more complex re-representations involving larger changes in structure are also possible (Yan et al., 2003).

Acting on all available re-representation opportunities can both be computationally expensive and potentially result in analogies that are meaningless or misleading, *e.g.*, "analogical hallucinations" (Gentner & Kurtz, 2006, Page 616). There are many possible strategies for selecting which re-

representations to perform. Two general principles underlie all such strategies (Yan et al., 2003, Page 1269):

1. **Systematicity**: All else being equal, re-representation suggestions that lead to increases in the systematicity of the derived analogy will be preferred.
2. **High Selectivity**: The selection process should be tightly controlled, so that very few of the possible opportunities are selected for consideration.

The above considerations yield the following computational-level models of representation under SMT. All three models assume that re-representation is done to improve on a given optimal analogy. The first of these models is general, in that it does not require that the created analogy be an extension of the given analogy.

ANALOGY DERIVATION WITH RE-REPRESENTATION (ADR)
Input: Predicate-structures B and T, an optimal analogy $A(B,T)$, a rule-set R, and integers k and c.
Output: Predicate-structures B' and T' and an analogy $A'(B',T')$ such that (i) B' and T' are derivable from B and T by at most k applications of rules from R and (ii) $val(A') - val(A) \geq c$.

The second and third models are restrictions of the first, as they are required to extend the given analogy.

ANALOGY IMPROVEMENT WITH RULE-GUIDED RE-REPRESENTATION (AIR[R])
Input: Predicate-structures B and T, an optimal analogy $A(B,T)$, a rule-set R, and integers k and c.
Output: Predicate-structures B' and T' and an analogy $A'(B',T')$ such that (i) $A \subset A'$, (ii) B' and T' are derivable from B and T by at most k applications of rules from R, and (iii) $val(A') - val(A) \geq c$.

ANALOGY IMPROVEMENT WITH CONTEXT-GUIDED RE-REPRESENTATION (AIR[C])
Input: Predicate-structures B and T, an optimal analogy $A(B,T)$, and integers k and c.
Output: Predicate-structures B' and T' and an analogy $A'(B',T')$ such that (i) $A \subset A'$, (ii) B' and T' are derivable from B and T by at most k context-guided re-representations, and (iii) $val(A') - val(A) \geq c$.

For simplicity, we will assume that all context-guided re-representations in the third model are of the basic "hole" type shown in part (b) of Figure 2.

It is possible that the act of analogy derivation rather than re-representation may artificially boost the difficulty of the models described above. To this end, we will also analyze a fourth model of re-representation, whose goal is to re-represent a given predicate-structure in order to satisfy a polynomial-time computable function *Prop* that returns either *True* or *False*, *e.g.*, does the re-represented T contain a particular type of easily-recognizable structure?

GENERAL DERIVATION WITH RE-REPRESENTATION (GDR)
Input: Predicate-structure T such that $Prop(T) = False$, rule-set R, and integer k.
Output: Predicate-structure T' such that (i) T' derivable by at most k applications of rules from R and (ii) $Prop(T') = True$.

The four models above are those that will be considered below. However, as will be explained later in the paper, results derived relative to these models have implications for a broad range of cognitive theories invoking re-representation.

Re-representation is Intractable

To investigate the computational (in)tractability of the models of re-representation given in the previous section, we have adopted standard complexity-theoretic proof techniques from Computer Science (Garey & Johnson, 1979). Using these techniques, we have proven the following (see the supplementary materials for proofs[1]):

Result 1 *ADR, AIR[R], AIR[C], and GDR are NP-hard.*

These results imply that there do not exist any algorithms for performing basic re-representation in the sense of the models considered here in polynomial time for all inputs (i.e., time upper-bounded by some function n^c where n is a measure of input size and c is some constant).[2] In other words, all algorithms for these models will run in exponential time or worse (i.e., time upper-bounded at best by some function c^n for c and n as above). As exponential-time algorithms have unrealistically long runtimes for all but very small inputs, they are generally considered to be computationally intractable (Garey & Johnson, 1979).

Given that it is *NP*-hard to derive analogies of a specified systematicity (van Rooij, Evans, Müller, Gedge, & Wareham, 2008; Veale & Keane, 1997), the *NP*-hardness of ADR is not unexpected. The *NP*-hardness of AIR[R] and AIR[C] is surprising, as deriving analogies that must be built on and include given analogies (Forbus, Ferguson, & Gentner, 1994) was not previously thought to be intractable. This suggests that the act of re-representation *all by itself* is intractable, which is confirmed by the *NP*-hardness of GDR. That all of these results hold in the most basic case as well – that is, re-representation of individual predicates — has additional power, as this means that these results may actually *under-estimate* the complexity of more complex types of re-representation invoking larger scale structural changes such as those proposed in Yan et al. (2003).

All this being said, the above does *not* say that re-representation is impossible – rather, it suggests that re-representation in practice may require one or more additional constraints on the inputs and/or the re-representation process

[1] `http://www.cs.mun.ca/~harold/Papers/ICCM12supp.pdf`

[2] This assumes that the conjecture $P \neq NP$ is true, which is widely believed within the Computer Science community on both theoretical and empirical grounds (Fortnow, 2009).

not considered so far in order to be computationally practical. In the next section, we describe a methodology that can be used to both model such specific constraints and investigate their computational effects.

A Method for Identifying Tractability Conditions

A computational problem that is intractable for unrestricted inputs may yet be tractable for non-trivial restrictions on the input. This insight is based on the observation that some *NP*-hard problems can be solved by algorithms whose running time is polynomial in the overall input size and non-polynomial only in some aspects of the input called **parameters**. In other words, the main part of the input contributes to the overall complexity in a "good" way, whereas only the parameters contribute to the overall complexity in a "bad" way. In such cases, the problem Π is said to be **fixed-parameter tractable** (Downey & Fellows, 1999) for that set of parameters. The following definition states this idea more formally.

Definition 1 *Let* Π *be a problem with parameters* $k_1, k_2, \ldots$. *Then* Π *is said to be* fixed-parameter (fp-) tractable *for parameter-set* $K = \{k_1, k_2, \ldots, k_c\}$ *if there exists at least one algorithm that solves* Π *for any input of size n in time* $f(k_1, k_2, \ldots, k_c)n^c$, *where* $f(\cdot)$ *is an arbitrary function and c is a constant. If no such algorithm exists then* Π *is said to be* fixed-parameter (fp-) intractable *for parameter-set K.*

In other words, a problem Π is fp-tractable for a parameter-set K if all superpolynomial-time complexity in solving Π can be confined to the parameters in K. In this sense the unbounded nature of the parameters in K can be seen as a reason for the intractability of the unconstrained version of Π. For any given fixed-parameter (in)tractability result, other results may be implied courtesy of the following lemmas:

Lemma 1 *If* Π *is fp-tractable for K then* Π *is fp-tractable for any* K' *such that* $K \subset K'$.

Lemma 2 *If* Π *is fp-intractable for K then* Π *is fp-intractable for any* K' *such that* $K' \subset K$.

It follows from the definition of fp-tractability that if an intractable problem Π is fp-tractable for parameter-set K, then Π can be efficiently solved even for large inputs, provided only that all the parameters in K are relatively small. In the next section we report on our investigation of whether or not parameters may be used in this way to render the models ADR, AIR[R], AIR[C], and GDR tractable.

What Does (and Doesn't) Make Re-representation Tractable?

Table 1 lists the parameters that we will consider in our fixed-parameter analyses of re-representation. Each of these parameters is of interest for different reasons. Parameters o and p are already known to individually render analogy derivation fp-tractable (van Rooij et al., 2008; Wareham, Evans, & van Rooij, 2011) and may in turn make analogy derivation with re-representation fp-tractable. Parameters k and $|R|$ explicitly and implicitly, respectively, encode the High Selectivity principle for re-representation selection strategies, and should thus be small in practice. Finally, in addition to considering parameters that separately characterize the inputs (o, p) and the re-representation process $(k, |R|)$, we will investigate parameter a, which in a sense encodes the degree of interaction between the given predicate-structures and the re-representation mechanisms (either rules in R or hole-contexts) in terms of the number of opportunities that these inputs provide for the application of these mechanisms.

Table 1: Overview of Parameters Considered.

Name	Description
o	Maximum number of objects over B and T
p	Maximum number of predicates over B and T
k	Amount of allowed re-representation
$\|R\|$	Rule-set size
a	Total number of re-representation opportunities in B and T

The results of our analyses relative to these parameters are given below (see the supplementary materials for proofs). As we are still in the early stages of our investigation, these results in tandem with Lemmas 1 and 2 do not yet fully characterize the parameterized complexity of our models relative to all possible combinations of the considered parameters. However, even at this initial stage, we can still draw some interesting conclusions and conjectures.

Let us start with the fp-intractability results:

Result 2 *ADR and GDR are fp-intractable for parameter-sets* $\{o, k, a\}$ *and* $\{k, |R|\}$.

Result 3 *AIR[R] is fp-intractable for parameter-set* $\{o, k, a\}$.

Result 4 *AIR[C] is fp-intractable for parameter-set* $\{o, k\}$.

Though there are still some open questions (in particular, the parameterized status of AIR[R] relative to $\{|R|\}$, AIR[C] relative to $\{a\}$, and GDR relative to $\{p\}$), these results in tandem with Lemma 1 establish that almost none of the parameters considered here can, if individually restricted to small values, render any of our models computationally feasible. The same also holds for any combinations of the parameters within the parameter-sets mentioned in these results. Of particular note is the fact that neither of the four models considered here can be made feasible by restricting k alone. This suggests that other principles in addition to High Selectivity must underlie re-representation selection strategies if re-representation is to be made feasible. These principles may have to be model-specific; for example, the current scarcity of fp-intractability results for AIR[R] and AIR[C] suggests that requiring derived analogies to build on given analogies may provide model-specific opportunities for restrictions that yield fp-tractability.

Consider now the fp-tractability results:

Result 5 *ADR, AIR[R], and AIR[C] are fp-tractable for parameter-set* $\{p\}$.

Result 6 *GDR is fp-tractable for parameter-set* $\{p, |R|\}$.

Result 7 *ADR and AIR[R] are fp-tractable for parameter-set* $\{o, |R|, a\}$.

Result 8 *AIR[C] is fp-tractable for parameter-set* $\{o, a\}$.

Result 9 *GDR is fp-tractable for parameter-set* $\{|R|, a\}$.

Each of these results implies that if all parameters in that result's parameter-set have small value, then the model mentioned in that result can be computationally feasible on inputs of arbitrary size. For example, Result 8 says that if o and a are simultaneously of small value, then AIR[C] may be computationally feasible. Results 7, 8, and 9 are of particular interest. The constraint on predicate-structure size imposed by o is not overly onerous, as many kinds of predicate-structures are based on a relatively small number of objects (Schlimm, 2008); moreover, it seems reasonable to conjecture that for certain applications (*e.g.*, those involving large-scale re-representation rules), a and $|R|$ may be suitably small.

Generality of Results

All of the intractability results reported in this paper, though defined relative to a specific theory of analogy derivation, have broad applicability. This is because the models examined here are restricted versions of models for other cognitive theories that invoke re-representation, *e.g.*,

- The re-representation modes encoded in our models are used in many cognitive theories (*e.g.*, GDR's single-structure re-representation parallels re-representation in insight problem solving (Ohlsson, 1992)).
- The predicate-structures on which our models are based are a powerful but basic form of representation, and it seems reasonable to conjecture that these other theories can be phrased in terms of predicate-structures.
- The basic single-predicate-change rules and hole-contexts used in our models are special cases of the more complex re-representation invoked in these other theories.

Results for models of other theories that satisfy the above then follow from the well-known observation that intractability results for a problem Π also hold for any problem Π' that has Π as a special case and can hence solve Π (suppose Π is intractable; if Π' is tractable, then it can be used to solve Π efficiently, which contradicts the intractability of Π – hence, Π' must also be intractable).

Our fp-tractability results are more fragile, as innocuous changes in the form of the inputs or the re-representation rules and contexts may in fact violate assumptions critical to the operation of the algorithms underlying these results. For now, we can say that as the parameters mentioned in Results 7, 8, and 9 encode only the combinatorics of re-representation possibilities (via $|R|$ and/or a) and require only that the structures generated by each such possible set of re-representations can be evaluated to determine if they comprise a viable solution in a reasonable amount of time, these results apply to all models whose input-types and re-representation mechanisms satisfy these conditions.

Discussion

Our research was motivated by the question of how the computational difficulty of re-representation in general can be reconciled with the ease of many instances of everyday re-representation. To address this question, we first set out to assess using formal methods whether re-representation as proposed in one such instance, namely analogy derivation, was computationally tractable. We found that this is not the case. In contrast, even these models of analogy derivation that only allow the simplest forms of re-representation can be proven *NP*-hard (Result 1). This means that no practical (read: polynomial time) algorithm can exist that perform such re-representation for all representations. To our knowledge, this is the first formal proof of the intractability of re-representation in the context of analogy derivation.

As this left the questions of how and under what conditions re-representation can become tractable, we performed complexity analyses to identify parameters that when restricted to small values render re-representation tractable (see Table 1 and its associated section for results). We believe that the following two of our findings are of particular interest:

1. Limiting the amount of re-representation (i.e., small k) does not by itself (nor when combined with many other parameters) make re-representation tractable (Results 2–4).
2. What does make re-representation tractable in the case of analogy (and, as noted above, many other more complex models) is when all of the parameters in the sets $\{p\}$ (Result 5) or $\{o, |R|, a\}$ (Results 7 and 8) are simultaneously restricted to take small values.

The latter set in (2) may be applicable to re-representation in everyday analogy derivation (especially those cases applying large-scale re-representations) and the former set may be reasonable for re-representation in concept development, as it is strongly hypothesized that children's representations are object- and attribute-rich and relationally poor (i.e., small p) (Gentner et al., 1995). The question now is whether these properties actually hold in these and other observedly fast forms of re-representation. If empirical evidence of these properties can be found, then our tractability results provide a psychologically plausible explanation of how the modelled forms of re-representation can be tractable despite the intractability of re-representation in general.

To summarize, in this paper we have given the first formal proofs not only that re-representation is computationally difficult even by itself, but that there are restrictions that may allow it to operate quickly in practice. Promising directions for future work include extending parameterized analyses of the models defined here to other parameters (in particular, parameters like a that describe interactions between the given input and the re-representation process), developing good fixed-parameter algorithms for re-representation within analogy derivation for implementation in large-scale AI systems like the Companions architecture (Forbus & Hinrichs, 2006), and investigating in detail the extent to which results and conclusions presented here apply to other models of re-representation-assisted analogy such as AMBR (Kokinov & Petrov, 2000) and HDTP (Krumnack et al., 2008) as well as models of insight problem solving and creativity.

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References

Downey, R., & Fellows, M. (1999). *Parameterized Complexity*. Berlin: Springer.

Forbus, K. D., Ferguson, R. W., & Gentner, D. (1994). Incremental structure-mapping. In *Proceedings of the 16th Annual Conference of the Cognitive Science Society* (pp. 313–318). Mahweh, NJ: Lawrence Erlbaum Associates.

Forbus, K. D., & Hinrichs, T. (2006). Companion Cognitive Systems: A Step Towards Human-level AI. *AI Magazine*, *27*(2), 83–95.

Fortnow, L. (2009). The Status of the P Versus NP Problem. *Communications of the ACM*, *52*(9), 78–86.

Garey, M. R., & Johnson, D. S. (1979). *Computers and Intractability: A Guide to the Theory of NP-Completeness*. San Francisco, CA: W.H. Freeman.

Gentner, D. (1983). Structure-Mapping: A Theoretical Framework for Analogy. *Cognitive Science*, *7*, 155–170.

Gentner, D., Brem, S., Ferguson, R., Wolff, P., Markman, A. B., & Forbus, K. D. (1997). Analogy and creativity in the works of Johannes Kepler. In T. B. Ward, S. M. Smith, & J. Vaid (Eds.), *Creative thought: An investigation of conceptual structures and processes* (pp. 403–459). Washington, DC: American Psychological Association.

Gentner, D., & Kurtz, K. J. (2006). Relations, Objects, and the Composition of Analogies. *Cognitive Science*, *30*, 609–642.

Gentner, D., Rattermann, M. J., Markman, A., & Kotovosky, L. (1995). Two forces in the development of relational similarity. In T. J. Simon & G. S. Halford (Eds.), *Developing cognitive competence: New approaches to process modelling* (pp. 263–313). Mahweh, NJ: Lawrence Erlbaum Associates.

Holyoak, K. J., & Thagard, P. (1989). Analogical mapping by constraint satisfaction. *Cognitive Science*, *13*, 295–355.

Kokinov, B. N., & Petrov, A. (2000). Dynamic Extension of Episode Representation in Analogy-Making in AMBR. In *Proceedings of the 22nd Annual Conference of the Cognitive Science Society* (pp. 274–279). Mahweh, NJ: Lawrence Erlbaum Associates.

Krumnack, U., Gust, H., Kühnberger, K.-U., & Schwering, A. (2008). Re-representation in a Logic-Based Model for Analogy Making. In W. Wobcke & M. Zhang (Eds.), *AI 2008* (pp. 42–48). Berlin: Springer-Verlag.

Kurtz, K. J. (2006). Re-representation in comparison: building an empirical case. *Journal of Experimental & Theoretical Artificial Intelligence*, *17*(4), 447–459.

Ohlsson, S. (1992). Information-processing explanations of insight and related phenomena. In M. Keane & K. Gilhooly (Eds.), *Advances in the Psychology of Thinking* (pp. 1–44). London: Harvester-Wheatsheaf.

Schlimm, D. (2008). Two Ways of Analogy: Extending the Study of Analogies to Mathematical Domains. *Philosophy of Science*, *75*(2), 178–200.

van Rooij, I., Evans, P., Müller, M., Gedge, J., & Wareham, T. (2008). Identifying Sources of Intractability in Cognitive Models: An Illustration using Analogical Structure Mapping. In *Proceedings of the 30th Annual Conference of the Cognitive Science Society* (pp. 915–920). Austin, TX: Cognitive Science Society.

Veale, T., & Keane, M. T. (1997). The Competence of Sub-Optimal Theories of Structure Mapping on Hard Analogies. In *Proceedings of the 15th International Joint Conference on Artificial Intelligence (IJCAI'97)* (Vol. 1, pp. 232–237). San Mateo, CA: Morgan Kaufmann.

Wareham, T., Evans, P., & van Rooij, I. (2011). What Does (and Doesn't) Make Analogical Problem Solving Easy? A Complexity-Theoretic Investigation. *Journal of Problem Solving*, *3*(2), 30–71.

Welling, H. (2007). Four Mental Operations in Creative Cognition: The Importance of Abstraction. *Journal of Creativity Research*, *19*(2–3), 163–177.

Winston, P. H. (1980). Learning and reasoning by analogy. *Communications of the ACM*, *23*(12), 689–703.

Yan, J., Forbus, K. D., & Gentner, D. (2003). A theory of rerepresentation in analogical matching. In *Proceedings of the 25th Annual Conference of the Cognitive Science Society* (pp. 1265–1270). Mahweh, NJ: Lawrence Erlbaum Associates.