Computer Science 6789 (Winter 2014):
Assignment #2
Due: 10:30 AM on Thursday, March 6, 2014

1. **(25 marks)** Given the decision problems

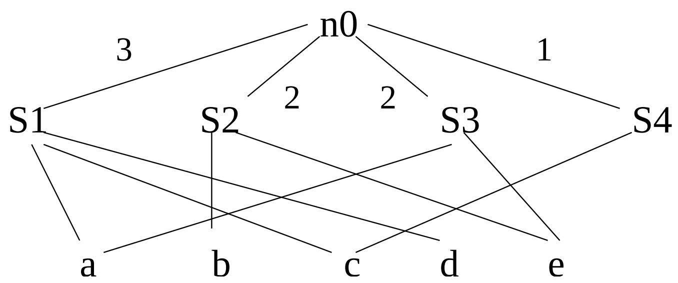
   Exact Set Cover (ESC)
   Input: A base-set $U = \{u_1, u_2, \ldots, u_n\}$ and a set $S = \{S_1, S_2, \ldots, S_m\}$ of subsets of U.
   Question: Is there a subset S' of the sets in S such that the sets in S' are disjoint, *i.e.*, for any two sets $S_i, S_j \in S'$, $S_i \cap S_j = \emptyset$, and the union of the sets in S' is U?

   Steiner Tree in Weighted Graphs (STG)
   Input: A weighted undirected graph $G = (V, E, w)$, a set $V' \subseteq V$, and an integer $k > 0$.
   Question: Is there a subtree $T = (V_T, E_T)$ of G such that $V' \subseteq V_T$ and $\sum_{e \in E_T} w(e) \leq k$?

   consider the following proposed many-one polynomial-time reduction:

   > **ESC** $\propto$ **STG**: Given an instance $\langle U, S \rangle$ of ESC, construct the following instance $\langle G = (V, E, w), V', k \rangle$ of STG: Let $V = \{n_0\} \cup \{S_j\} \cup \{u_i\}$, $E = \{(n_0, S_j)\} \cup \{(S_j, u_i) \mid u_i \in S_j\}$, $w((n_0, S_j)) = |S_j|$ and $w((S_j, u_i)) = 0$, $V' = \{n_0\} \cup \{u_i\}$, and $k = |U|$.

   For example, given an instance of ESC in which $U = \{a, b, c, d, e\}$ and $S = \{S_1, S_2, S_3, S_4\}$ where $S_1 = \{a, c, d\}$, $S_2 = \{b, e\}$, $S_3 = \{a, e\}$, and $S_4 = \{c\}$, the reduction produces the following instance of STG (note that all edges without explicitly stated weights have weight 0):



   Observe that the given instance of ESC has the solution $\{S_1, S_2\}$ which corresponds to the tree T in the constructed instance of STG with $V_T = \{n_0, S_1, S_2, a, b, c, d, e\}$, $E_T = \{(n_0, S_1), (n_0, S_2), (S_1, a), (S_1, c),$
   $(S_1, d), (S_2, b), (S_2, e)\}$, and $\sum_{e \in E_T} w(e) = 3 + 2 + 0 + 0 + 0 + 0 + 0 = 5 = |U|$.

   (a) **(10 marks)** Prove that this reduction is incorrect by giving a counterexample such that the given instance of ESC and the constructed instance of STG have different answers.

(b) **(15 marks)** Give a many-one polynomial time reduction from the following problem to STG:

MINIMUM SET COVER (MSC)
Input: A set $U = \{u_1, u_2, \ldots, u_n\}$, a set $S = \{S_1, S_2, \ldots, S_m\}$ of subsets of U, and an integer k.
Question: Is there a subset S' of S such that $|S'| \leq k$ and the union of the sets in S' is U?

Briefly discuss why your reduction is correct by showing that a given instance of MSC has answer "yes" if and only if the instance of STG constructed by the reduction from this given instance has answer "yes".

2. **(20 marks)** Prove the following:

(a) **(10 marks)** Fixed-paramneter reducibility is transitive, *i.e.*, if $\{k\}$-**X** fp-reduces to $\{k'\}$-**Y** and $\{k'\}$-**Y** fp-reduces to $\{k''\}$-**Z** then $\{k\}$-**X** fp-reduces to $\{k''\}$-**Z**.

(b) **(10 marks)** If $\{k_1, k_2, \ldots, k_m\}$-**X** fp-reduces to $\{k'_1, k'_2, \ldots, k'_n\}$-**Y** and $\{k'_1, k'_2, \ldots, k'_n\}$-**Y** is fp-tractable then $\{k_1, k_2, \ldots, k_m\}$-**X** is fp-tractable.

3. **(75 marks)** For each of the following parameterized problems, either prove that this problem is fixed-parameter tractable (by giving an fp-tractable algorithm) or prove that this problem is fp-intractable (by showing that it is either hard for or cannot be included in some class in the W-hierarchy above FPT). In the latter case, include reductions where appropriate and state any problem-class-relationships, *e.g.*, $FPT \neq W[1]$, that are required to prove fp-intractability.

(a) **(15 marks)**

$\{k\}$-DOMINATING l-CLIQUE-SET
Input: An undirected graph $G = (V, E)$ of maximum vertex-degree d and a positive integers k and l, $k > 0$ and $l \geq 3$.
Question: Does G have a dominating l-clique-set of size k, *i.e.*, is there a set of k cliques of size l in G such that for every clique c in G of size l, either c is in this subset or some vertex in c is connected by an edge to a vertex in some clique in this subset?

(b) **(15 marks)**

$\{l\}$-DPCLIQUE
Input: An undirected graph $G = (V, E)$ and positive integers k and l.
Question: Is there a subset $V' \subseteq V$ such that $|V'| \geq k$, there is a simple path in G between each pair of vertices in V' that contains at most l edges, and all of these paths are disjoint, *i.e.*, none of the simple paths between any two pairs of vertices in V' share an edge?

(c) **(15 marks)**

$\{d, k\}$-HYPERGRAPH c-COVER
Input: A set of vertices V, a set E of subsets of V of size at most $d \geq 3$, and positive non-zero integers k and c.
Question: Is there a subset $V' \subseteq V$, $|V'| \leq k$, such that for each $e \in E$, $|e \cap V'| \geq c$?

(d) **(15 marks)**

$\{l\}$-SIMPLE PATH NUMBER
Input: A cubic undirected graph $G = (V, E)$, a pair of vertices u and v in G, and and a positive integer l.
Question: Is the number of simple paths of length $\leq l$ between u and v in G at most k?

A graph is cubic if each vertex in the graph has degree three.

(e) **(15 marks)**

$\{k\}$-STEINER SUBGRAPH
Input: An undirected graph $G = (V, E)$, a subset $S \subset V$, and a positive integer k.
Question: Is there a subset $T \subset V - S$, $|T| \leq k$, such that the graph $G' = (S \cup T, \{(u, v) : u, v \in S \cup T \text{ and } (\text{u}, \text{v}) \in \text{E}\})$ is connected?

Bonus Questions:

B1: (15 marks) What is the parameterized complexity of $\{k, l\}$-DOMINATING l-CLIQUE-SET? Give a proof of your answer.

B2: (15 marks) What is the computational complexity of SIMPLE PATH NUMBER? Give a proof of your answer.