1. **(14 marks)** Determine the optimal parenthesization of a matrix-chain product whose sequence of dimensions is \((2, 5, 3, 4, 2, 3, 5)\) using the algorithms given on page 375 and 377 of the textbook. In the style of Figure 15.5, show the filled-in dynamic programming matrices \(m\) and \(s\), the “backpointer path” in \(s\) that gives an optimal parenthesization, and the parenthesization associated with that path.

2. **(20 marks)** Consider the following edge-weighted directed graph:

![Graph Image](image)

   a) **(10 marks)** Run Dijkstra’s algorithm (p. 658) on the directed graph above using vertex \(x\) as the source vertex. In the style of Figure 24.6 in the textbook, show the \(d\) and \(\pi\) values and the vertices in set \(S\) after each iteration of the while loop.

   b) **(10 marks)** Run the Bellman-Ford algorithm (p. 651) on the directed graph above using vertex \(x\) as the source vertex. Relax edges in lexicographic order in each pass, and in the style of Figure 24.4 on the textbook, show the \(d\) and \(\pi\) values after each pass. Finally, give the boolean value returned by the algorithm.

3. **(16 marks)** Consider the following decision problems:

**Dumbbell subgraph (DS)**

*Input:* An undirected graph \(G = (V, E)\) and two positive integers \(k, l \geq 1\).

*Question:* Are there two cliques \(C_1\) and \(C_2\) and a simple path \(P\) in \(G\) such that \(C_1\) and \(C_2\) have \(\geq k\) vertices apiece, \(P\) has \(\geq l\) edges, \(P\) connects \(C_1\) and \(C_2\), the cliques and path do not have any edges in common, and the only vertices that \(P\) shares with \(C_1\) (\(C_2\)) is its connection-vertex?
**Bounded-Weight Subset Cover (BWSSC)**

*Input:* A set $I = \{i_1, \ldots, i_a\}$ of items, a set $R = \{r_1, \ldots, r_b\}$ of subsets of $I$, an integer-valued subset-weight function $w()$ such that for each $r_x \in R$, $w(r_x) > 0$, a subset $N \subseteq I$, and integers $0 < k_1 \leq k_2$.

*Question:* Is there is a subset $R' \subseteq R$ such that $\cup r \in R' r = N$ and $k_1 \leq \sum_{r \in R'} w(r) \leq k_2$?

a) **(8 marks)** Prove that problem DS is $NP$-complete by (1) showing that this problem is in $NP$ and (2) giving a polynomial-time many-one reduction (algorithm + proof of correctness) to the following $NP$-hard problem:

**Hamiltonian Path (HP)**

*Input:* An undirected graph $G$.

*Question:* Is there a simple path in $G$ that links all vertices in $G$?

b) **(8 marks)** Prove that problem BWSSC is $NP$-complete by (1) showing that this problem is in $NP$ and (2) giving a polynomial-time many-one reduction (algorithm + proof of correctness) to this problem from the following $NP$-hard problem:

**Set Cover (SC)**

*Input:* A set $U = \{u_1, \ldots, u_n\}$ of items, a set $S = \{s_1, \ldots, s_m\}$ of subsets of $U$, and an integer $k \leq m$.

*Question:* Is there a subset $S' \subseteq S$ such that $|S| \leq k$ and $\cup_{s \in S'} s = U$?