Fraunhofer Institut IIE, 1994, pp. 66 — 73

A Dynamical Implementation of Self-organizing Maps

Wolfgang Banzhaf *

Department of Computer Science, University of Dortmund, Germany

and

Manfred Schmutz †

Fraunhofer Institut für Produktionstechnik und Automatisierung,

Stuttgart, Germany

Abstract

The standard learning algorithm for self-organizing maps (SOM) involves the two steps of a search for the best matching neuron and of an update of its weight vectors in the neighborhood of this neuron. In the dynamical implementation discussed here, a competitive dynamics of laterally coupled neurons with diffusive interaction is used to find the best-matching neuron. The resulting neuronal excitation bubbles are used to drive a Hebbian learning algorithm that is similar to the one Kohonen uses. Convergence of the SOM is achieved here by relating time (or number of training steps) to the strength of the diffusive coupling. A standard application of the SOM is used to demonstrate the feasibility of the approach.

1 Introduction

One of the strengths of neural networks is their capability of distributed processing of information. Numerous authors have been stressing this point over the years [1, 2]. Many arguments have been brought forward against localized representation of information in so-called grandmother cell neurons which would take with them, if destroyed, all the knowledge they have accumulated over extended periods of training. As long as neural networks are simulated using traditional computers, however, there is basically no need to require the degree of fault-tolerance brought about by a distributed representation. Indeed, a distributed representation might even be less efficient in applications, since it requires the use of many neurons — and therefore simulation cpu-cycles — to represent a pattern. It is only in hardware-based neural networks and reality-based brains that implementation of a distributed representations becomes evidently advantageous.

One example of a distributed representation for information is the topological feature map proposed by Kohonen [3] - [5] for the projection of high-dimensional pattern data into a low-dimensional feature space. Kohonen's algorithm is able to generate an ordered projection of patterns in a low-dimensional space during training by updating a neighborhood of the best-matching cell of a topologically organized lattice of artificial neurons. The process of ordering an initially random map is called self-organization. Its convergence has been shown to depend on the specific updating rule used. Once the map has been formed, a destruction of cells does not seriously perturb the behaviour of the network. Every incoming pattern may evoke a response in the entire neighborhood that was trained for it. Thus, even if the main motivation behind the self-organizing map (SOM) is the topological ordering of pattern projections, fault-tolerance is one of its immediate benefits.

The synergetic computer advocated by Haken [6] - [8], on the other hand, takes a classical task of pattern recognition into the realm of dynamical systems: If we represent patterns by real valued feature vectors, then their translation into a symbol stating their class membership is accomplished by a competitive dynamics between "grandmother cell neurons" that become responsible for a group of similar patterns, i.e. for a class. The dynamics was derived from natural competitive systems and was implemented by a specific interaction between cells as a winner-take-all (WTA) network. This dynamics has a natural time scale, characterized by the time needed to relax into one of N equilibrium states. If one sets up a competitive learning process using this kind of network, another time scale emerges, namely that of the competitive learning dynamics [9]. The existence of intrinsic time

*banzhaf@tarantoga.informatik.uni-dortmund.de, Baroper Str. 301, 44221 Dortmund, Germany

†Nobelstr. 12, 70569 Stuttgart, Germany

scales here is in remarkable contrast to most of the static learning algorithms employed in the literature and constitutes, in the oppinion of the authors, one of the main strengths of dynamical WTA networks, a strength one should build upon.

Conventional competitive learning proceeds by waiting for the relaxation state of the network to appear before updating connections. It is, however, interesting to observe the network learning successfully even when an unrelaxed state is used during update of connections. It is this parameter region that we have called non-equilibrium learning [9] where flexibility and performance are at their best. In the present contribution we shall apply the competitive dynamics introduced by Haken to the formation of a SOM, by adding an appropriate extension to the original dynamics. In a recent paper we have introduced this concept as a diffusive Haken dynamics [10].

The standard learning algorithm for self-organizing maps (SOM) involves two steps: a) a search for the feature neuron whose weight vector shows best match with the presented pattern; b) an update of all weight vectors in its neighborhood in the direction given by the presented pattern. Stabilization of weight vectors is achieved by decreasing a parameter specifying the neighborhood size. Then, during the recognition phase, only step a) is used.

In the dynamical implementation discussed here, a competitive dynamics of laterally coupled neurons is used to find the best-matching neuron. The essential extension to Haken's original model is the substitution of its point-like relaxation states by excitation bubbles around the winning neuron. It has been shown that such a dynamics can be implemented by adding a diffusive interaction to the winner-take-all dynamics of the Haken-model. Step b) above is realized by a Hebbian learning rule similar to the one Kohonen [4] uses. In the current implementation, convergence is achieved by controlling the strength of the diffusive coupling with a quantity related to elapsed training time or number of training steps.

Section 2 will provide a short summary of the results of previous work concerning the diffusive Haken model, with special emphasis on practically relevant results. Section 3 then discusses the application of the diffusive Haken model to implement Kohonen's SOM. There, we shall discuss an appropriate learning rule and demonstrate its performance in Section 4.

2 The diffusive Haken model

Inspired by the analogy between pattern formation in synergetic systems and pattern recognition, Haken [6] has formulated a prototype dynamics for the implementation of the winner-take-all function. In this model, the states $\mathbf{q} \in R^N$ of the system are driven by a potential $V^{(1)}$

$$\dot{\mathbf{q}} = -\nabla V^{(1)}(\mathbf{q}) \tag{1}$$

with

$$V^{(1)} = -\frac{1}{2}D + \frac{1}{2}D^2 - \frac{1}{4}\sum_i q_i^4, \tag{2}$$

$$D = \sum_i q_i^2. \tag{3}$$

In exact terms, the equations of motion

$$\dot{q}_i(t) = (1 - 2D(t))q_i(t) + q_i^3(t) \tag{4}$$

may be interpreted as the dynamics of the excitation $q_i(t)$ of a network of N cells.

It can be shown that aside from a finite set of $2^N - N$ non-attracting stationary states, each initial state $q(t_0) \in R_+^N = \{\mathbf{q} \in R^N \mid q_i \geq 0\}$ will end up in one of N attractors which coincide with the corners of the unit hypercube $(1, 0, ..., 0), (0, 1, ..., 0), (0, 0, ..., 1)$. In more detail, an initial state $\mathbf{q}(t_0)$ with its maximal excitation in component $i_0, (q_{i_0}(t_0) > q_{i \neq i_0}(t_0))$ ends up in the attractor with $q_{i_0} = 1$, $q_{i \neq i_0} = 0$ (for the results of a simulation see Figure 1 a,b).

In this way, the Haken model performs a dynamical parallel search for the maximum in a set of data. In agreement with the very aims of the construction, the result is given in form of a strictly localized state where only a single cell (degree of freedom) is excited.

The winner-take-all function is an essential step for decision making in many pattern recognition applications. There are some applications, however, where the competition should lead to a bubble of excitation centered around the maximum of the input data (see Figure 1 c,d). As it is shown in more detail in [10] such behaviour can be achieved by adding a diffusive term to the potential $V^{(1)}(\mathbf{q})$ of the Haken model:

$$V^{(2)}(\mathbf{q}) = \frac{1}{2n}\alpha \sum_{\langle i,j \rangle}(q_i - q_j)^{2n}, \tag{5}$$

where n and $\alpha \geq 0$ are the order and coupling constant of the diffusive interaction. The bracket indicates summation over pairs of nearest neighbors.

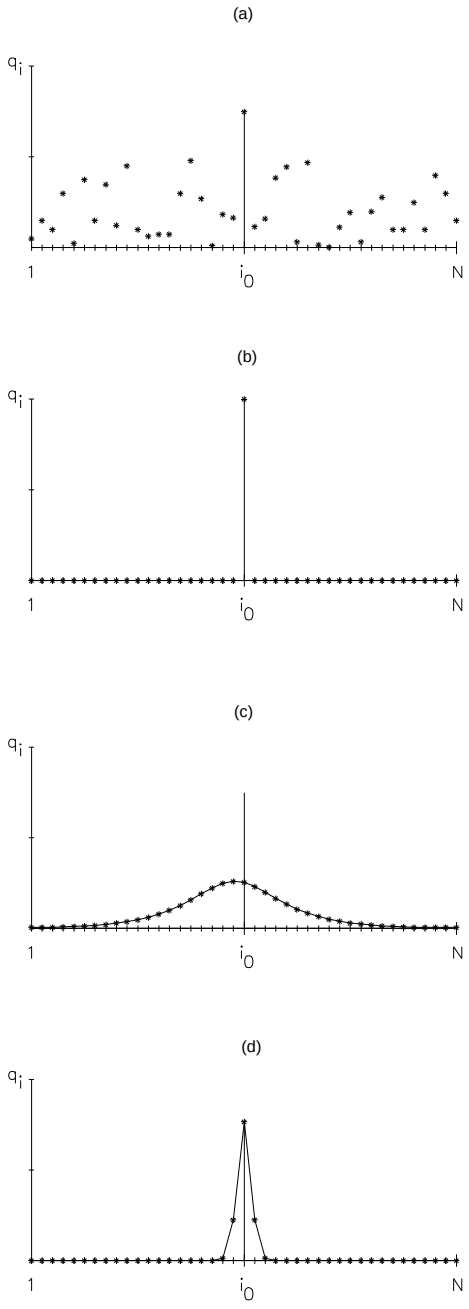

Figure 1: Original Haken model, (a) Initial distribution of cell activities, (b) ground state (relaxation state of the network). Modified model with additional diffusive interaction, ground state (relaxation state of the network) in one-dimensional topology, (c) standard diffusive interaction ($n = 1$), (d) non-standard diffusive interaction ($n = 2$).

Hence, the diffusive Haken model is defined in terms of the potential

$$V(\mathbf{q}) = V^{(1)}(\mathbf{q}) + V^{(2)}(\mathbf{q}) \tag{6}$$

which reduces to the original Haken model for $\alpha = 0$.

Whereas in the original model the numbering of cells is completely irrelevant, the definition of a short-range interaction of the form of equ. (5) predetermines a topological ordering of cells. A natural choice for the topology is a square lattice in d dimensions with periodic boundary conditions. As a result, the diffusive interaction brings into play aspects of topology and dimension. Together with the additional parameters α and n the model now obtains a rich structure and cannot any more be treated in a straightforward analytical manner.

Because of the numerous parameters, an analysis of the of ground states (relaxation states) of the network by means of numerical simulation has to be restricted to a few typical cases. Of foremost interest is the study of standard and second order non-standard diffusive interaction ($n = 1, 2$) in one and two dimensions ($d = 1, 2$). For a given set of parameters, the attractor state (unique up to translations) may be obtained as the asymptotic state evolving from a randomly chosen initial state. Figure 1 and 2 show examples for $d = 1$ and $d = 2$.

A systematic examination yields the following results:
(i) $\mathbf{d = 1}$: As anticipated, the diffusive interactions with $n = 1, 2$ have a broadening effect on the ground states which becomes more and more pronounced the more α is increasing. We interpret this behavior as striking a balance between the localizing force of the Haken potential $V^{(1)}(\mathbf{q})$ and the delocalizing force of the diffusion $V^{(2)}(\mathbf{q})$ which can be tuned by the coupling α. Experiments show that the delocalization brought about by $n = 1$ diffusion is much larger than that brought about by $n = 2$ diffusion.
(ii) $\mathbf{d = 2}$: In two dimensions even an arbitarily weak standard diffusive interaction delocalizes the ground state completely (see Figure 2b). In contrast, second order diffusive interaction still allows ground states of tunable size similar to the situation in $d = 1$.

A major obstacle to an analytic treatment of the model is its discrete nature. The situation becomes much more favorable when we replace the discrete cells by an excitable continuous medium, thus making the model accessible for the application of various field-theoretical methods.

Proceeding along these lines, the instanton technique gives the exact form of the ground states in

one dimension for $n = 1$ and $n = 2$. Furthermore, it turns out that important ground state properties can be determined astonishingly well by a simple variational calculation based on Gaussian test functions.

In summary, we obtain simple analytical expressions from the application of field theory. Characteristic quantities of ground states that depend on the coupling strength α such as their amplitude and halfwidth have been examined and given in [10] for all cases of interest. These expressions have been found to be reasonable approximations for the behavior of the discrete model as well, except for a small range of parameters near $\alpha = 0$, where the analogy between field theory and the discrete model breaks down.

3 Application of the diffusive Haken model

The Kohonen algorithm for the formation of topological maps is a competitive learning algorithm on a layer of neurons connected on a grid. It is a member of the class of vector quantization algorithms where patterns $\mathbf{x} = (x_1, ..., x_M)$ from pattern space R^M distributed according to the probability distribution $P(\mathbf{x})$ stimulate artificial neurons. Each neuron $i,\ i = 1, ..., N$, with activity or excitation q_i possesses a reference vector $\mathbf{w}_i \in R^M$ that determines its response to input stimulation.

Initially, N random reference vectors $\mathbf{w}_i(0),\ i = 1, ..., N$, are assigned to neurons. Self-organization is achieved by updating connections during a training phase at discrete times $t = 0, 1, 2, ...$ with

$$\mathbf{w}_i(t+1) = \mathbf{w}_i(t) + \epsilon(t)h(i, i_c)(\mathbf{x} - \mathbf{w}_i(t)) \qquad (7)$$

with $\epsilon(t)$ being a *time dependent learning parameter* which has to fulfill certain criteria in order to lead to convergence and $h(i, i_c)$ the so-called *neighborhood function*, usually taken to be Gaussian

$$h(i, i_c) = exp[-\frac{(i - i_c)^2}{2\sigma}] \qquad (8)$$

around the exitation peak at neuron i_c. σ is the width of the neighborhood and i_c is the index of that neuron whose reference vector matches $\mathbf{x}$ best, therefore carrying the highest activity.

Thus, the first step in the reference vector updating is always determination of the best matching cell. This is done by applying a predefined distance function $d_i = d_i(\mathbf{x}, \mathbf{w}_i)$ whose minimum

$$d_{i_c} = \min_{i \in 1,...,N} d_i(\mathbf{x}, \mathbf{w}_i) \qquad (9)$$

Figure 2: Modified model with additional diffusive interaction, ground state (relaxation state of the network) in two-dimensional topology. (a) Initial distribution of cell activities, (b) standard diffusive interaction ($n = 1$), (c) non-standard diffusive interaction ($n = 2$).

then forms the center of $h(i, i_c)$.

d_i could be related to a similarity measure for patterns like the scalar product, or it could be a distance measure like the Eucledian distance we use here

$$d(\mathbf{x}, \mathbf{w}_i) = \sum_j (x_j - w_{ij})^2. \tag{10}$$

The update equation (7) may be considered a discrete version of a continuous weight dynamics

$$\dot{\mathbf{w}}_i(t) = \tau_2 q_i(t)(\mathbf{x} - \mathbf{w}_i(t)) \tag{11}$$

lasting for a time $T = 1$. Here, $q_i(t)$ lumps together temporal and spatial influence

$$q_i(t) = \epsilon(t) h(i, i_c). \tag{12}$$

With the diffusive Haken model, a dynamical implementation of Kohonen's algorithm is now straightforward: The sequential processes of best-match search and neighborhood updating are realized via parallel dynamics derived from equ. (6). To this end, the time-derivative of the potential is used

$$\dot{q}_i = -\tau_1 \frac{\partial V(\mathbf{q})}{\partial q_i} \tag{13}$$

with a shorter time scale $\tau_1 < \tau_2$.

In each training step, a pattern is fed into the neural grid, and the diffusive Haken dynamics of equ. (13) is initialized with excitations

$$q_i(t_0) = 1/d_i(\mathbf{x}, \mathbf{w}_i). \tag{14}$$

If $\frac{\tau_1}{\tau_2} \ll 1$ we can assume the shorter (q-)dynamics to have relaxed into its equilibrium state of a localized bubble of excitation before updating of reference vectors according to (11) occurs. Note that $q_i(t)$ in (11) is now identified with the excitation of neuron i and no longer needs to be defined by external parameters. To be more exact, the updating process also occurs during relaxation of $\mathbf{q}(t)$, but the influence of the relaxed state on the updating is overwhelming.

If, on the other hand, $\frac{\tau_1}{\tau_2} < 1$, non-equilibrium learning occurs, that is learning takes place mostly *during* the relaxation process of $\mathbf{q}(t)$.

By using a Hebb-like learning rule such as in equ. (11) we can emboss the relaxation state appearance to the neural network. In the present case, the relaxation state is a local bubble of excitation, inducing the formation of topological order in the network.

The decrease of neighborhood size often used in Kohonen-type algorithms for enforcement of stability and order can be mimicked by chosing a time-dependent diffusive coupling constant α. A simple ansatz is

$$\alpha(t) = \frac{1}{1 + t/\tau_3} \tag{15}$$

with $\tau_3 \gg \tau_2$.

4 Implementation of SOMs

We have simulated the formation of one-dimensional ordered maps using the standard example of stimulus distribution on a triangle [5]. It was confirmed that using equations (11) and (13) in connection with equs. (14) and (15), a dynamical formation of SOMs is achievable.

Figure 4 shows an initial state of a network of $N = 20$ neurons with one-dimensional neighborhood connections for diffusive distribution of signals. Here and in the following, we show the triangle with the placement of network cells that are originally located randomly. Also shown is a stimulus, together with the reponse of the cells before and after a defined number of steps in τ_1-units.

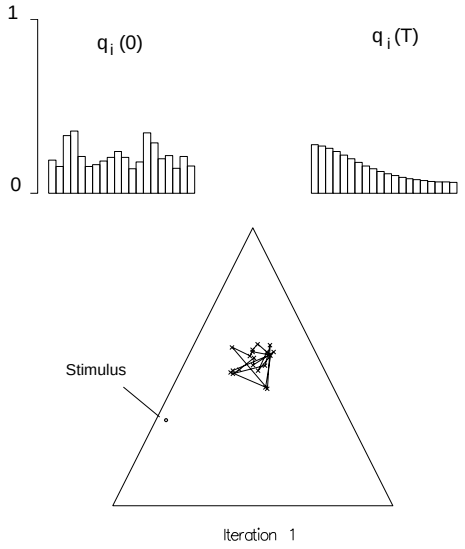

Figure 4: Diffusive Haken model with standard diffusion in $d = 1$. $N = 20, \alpha(0) = 1, \tau_2 = 10000\tau_1, \tau_3 = 1000\tau_2$. Initial distribution of cells and reponse to first stimulus before and after competitive dynamics (13).

In Figure 5, taken after 1000 training steps, an ordered network state has already appeared. Even before the competitive dynamics, the distribution of cells and their response is regular.

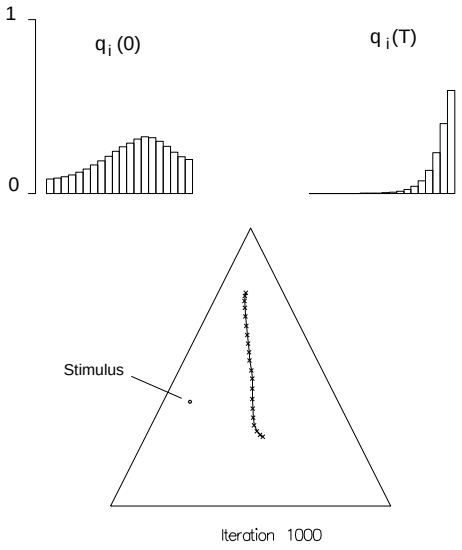

Figure 5: As before, after the first 1000 training steps (iterations) have passed.

Figure 6 shows the final state of the network (after convergence).

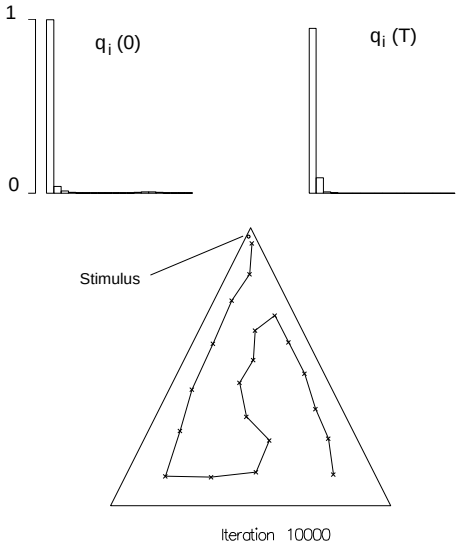

Figure 6: As before, after the network has stabilized within 10000 training steps.

For reference, we give the final distribution of cells if we apply 10000 stimuli to a network with non-standard diffusive interaction. The tendency towards more localized relaxation states (i.e. less width of the distribution) is obvious.

As mentioned before, it is not necessary to wait for the competitive dynumics to have approached the relaxation state before updating connections. Feasibility of this non-equilibrium case for learning is demonstrated with the next figures.

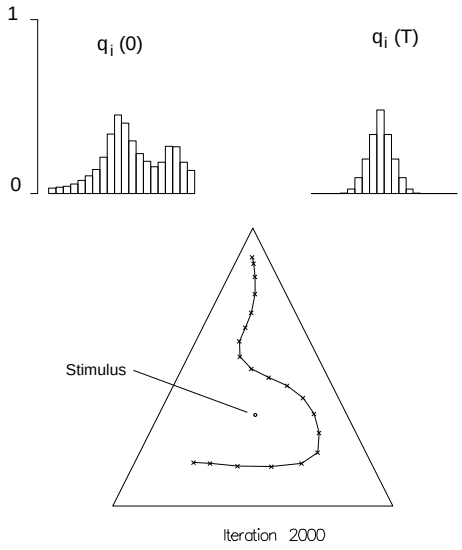

Figure 7: Diffusive Haken model with non-standard diffusion in $d = 1$. $N = 20, \alpha(0) = 50, \tau_2 = 10000\tau_1, \tau_3 = 500\tau_2$. Final distribution and reponse to stimulus before and after competitive dynamics.

Figure 8 shows again the initial stimulus to a network of $N = 20$ cells. This time, learning took place on a much faster time-scale, with an acceleration factor of 10000 for standard diffusion and of 1000 for the non-standard diffusion. The decay of α was larger by a factor 10, with the bulk of the accelaration coming from using only very few iterations through the competitive dynamics (13). We can see that an ordered state has barely arrived before updating connections.

In Figure 9, the network is already in an ordered state. Stimuli are more or less adjusted through the competitive dynamics. Slowly, a better distribution of cells is achieved.

After 10000 training steps the network has stabilized. Distribution is not as good as in the equilibrium case, but the solution was generated within 1/10000 of the time on a serial machine.

For reference, we give in a last Figure 11 the final state for a non-standard diffusion model after 20000 training steps.

5 Conclusions

It has been shown that a dynamical implementation of SOMs by using a modified version of Haken's

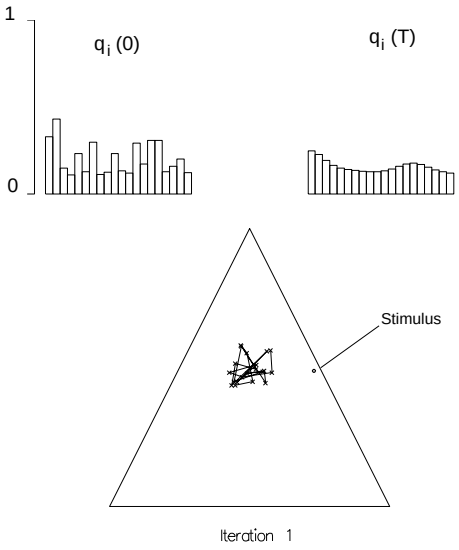

Figure 8: Diffusive Haken model with standard diffusion in $d = 1$. $N = 20, \alpha(0) = 1, \tau_2 = 10\tau_1, \tau_3 = 100\tau_2$. Initial distribution of cells and reponse to first stimulus before and after competitive dynamics for a very small number of iterations.

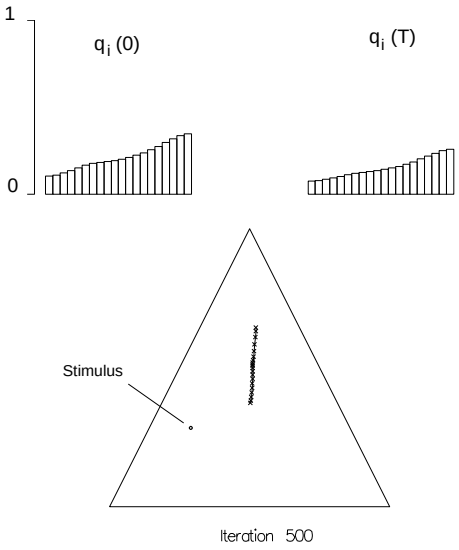

Figure 9: As before, intermediate status after 500 training steps (iterations) have passed.

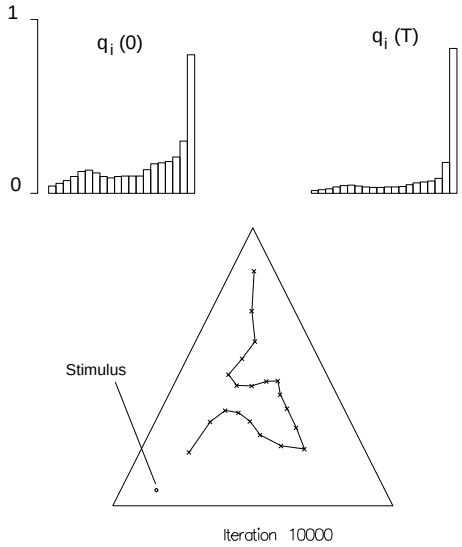

Figure 10: As before, final state after 10000 training steps.

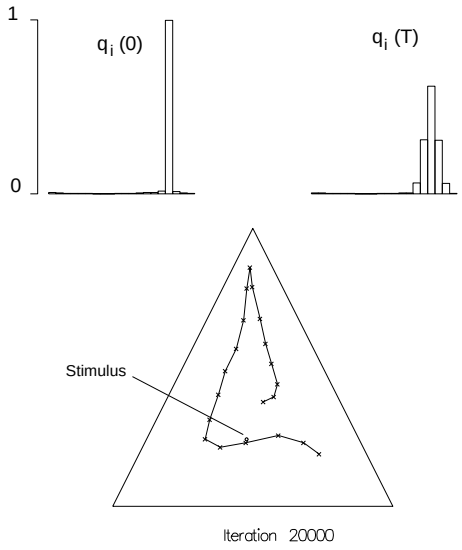

Figure 11: Diffusive Haken model with non-standard diffusion in $d = 1$. $N = 20, \alpha(0) = 50, \tau_2 = 100\tau_1, \tau_3 = 100\tau_2$. Final distribution and reponse to stimulus before and after competitive dynamics.

winner-take-all dynamics is possible. Analytical and numerical results for this modified model show that, depending on the particular order of diffusive interactions added to the original dynamics and depending on the dimension of the topology, either localized or de-localized relaxation states result. At present, only localized states have been used for SOM implementation.

The simulations shown above only comprise the case of a one-dimensional topology. There is no doubt, however, that in two dimensions with a correct choice of parameters, the model is able to implement SOMs. It is also suspected that models with de-localized relaxation states are useful for the formation of SOMs, especially in the non-equilibrium region of learning. Due to the diffusive interaction, ordered states are at least transients in the competitive dynamics. Since non-equilibrium learning makes use of transients anyway to arrive at ordered networks, also de-localizing diffusive interactions might be used.

Recently, Kohonen has pointed out [11] that his model can be imagined an algorithmic idealization of interactions in the brain. He considers non-synaptic interactions between neurons, mainly due to chemical messengers that diffuse over a neighborhood of cells, as serious candidates for an implementation of his algorithm. Again, diffusive interactions are necessary in order to arrive at the ordered state of the network. In brains, the topology would then be given by the three-dimensional spatial arrangement of neurons.

Over the last years, we have been examining some other extensions of Haken's dynamical winner-take-all network. A discussion of their features, as well as a more detailed account of the model presented here, has to be deferred to a later time.

References

[1] Rumelhart, D.E., McClelland, J., *Parallel Distributed Processing*, MIT Press, Cambridge, 1986

[2] Hopfield, J.J., *Neural Networks and and Physical Systems with Emergent Collective Computational Abilities*, Proc. Natl. Acad. Sci. USA **79** (1982) 2554

[3] Kohonen, T., *Clustering, taxonomy, and topological maps of patterns*, Proceedings of the 6th Int. Conf. on Pattern Recognition, Munich, Germany, 1982, 114 -128

[4] Kohonen, T., *Self-organized formation of topologically correct feature maps*, Biological Cybernetics **43** (1982) 59 - 69

[5] Kohonen, T., *Self-organization and Associative Memory*, Springer, Berlin, 1989, 3rd ed.

[6] Haken, H., *Synergetic computers for pattern recognition and associative memory*, in: Haken, H. (Ed.), *Computational systems, natural and artificial*, Proceedings of the Elmau Int. Symposion on Synergetics 1987, Springer, Berlin, 1987

[7] Haken H., *Nonequilibrium Phase Transitions in Pattern Recognition and Associative Memory*, Z.f.Phys. **B 70** (1988) 121 - 123

[8] Haken H., *Synergetic Computers and Cognition*, Springer, Berlin, 1991

[9] Banzhaf W. and Haken H., *Learning in a competitive network*, Neural Networks **3** (1990) 421 - 435

[10] Schmutz M. and Banzhaf W., *Robust competitive networks*, Physical Review **A45** (1992) 4132 - 4145

[11] Kohonen, T., *Physiological Interpretation of the Self-Organizing Map Algorithm*, Neural Networks **6** (1993) 895 - 905