Unit 3: Time Response, Part 2: Second-Order Responses

Engineering 5821:
Control Systems I

Faculty of Engineering & Applied Science
Memorial University of Newfoundland

January 28, 2010

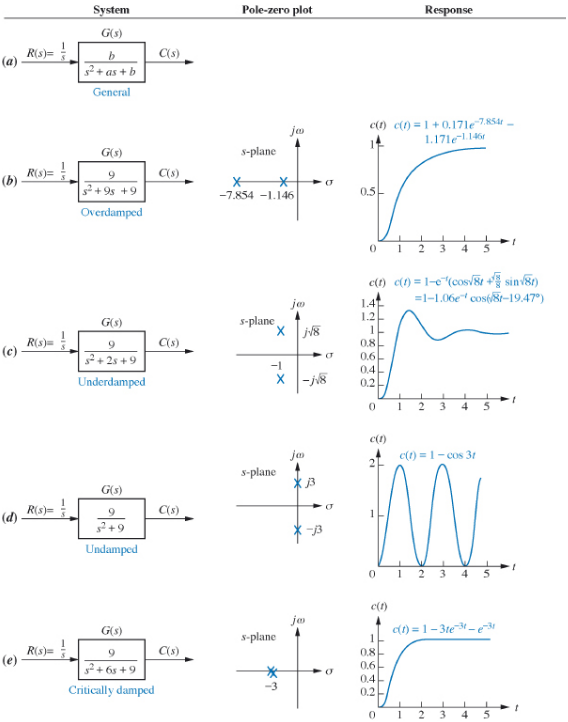
Second-Order Systems

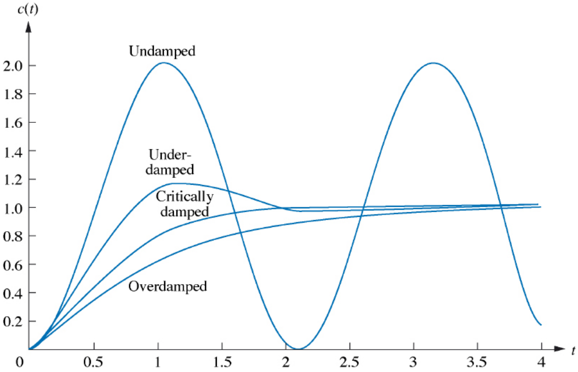
Second-order systems (systems described by second-order DE's) have transfer functions of the following form:

$$G(s) = \frac{b}{s^2 + as + b}$$

(This TF may also be multiplied by a constant K, which affects the exact constants of the time-domain signal, but not its form).

Depending upon the factors of the denominator we get four categories of responses. If the input is the unit step, a pole at the origin will be added which yields a constant term in the time-domain.


System
Pole-zero plot
Response
(a)
R(s)= 1/s
G(s)
b / (s^2 + as + b)
C(s)
General
(b)
R(s)= 1/s
G(s)
9 / (s^2 + 9s + 9)
C(s)
Overdamped
jω
s-plane
σ
−7.854 −1.146
c(t) c(t) = 1 + 0.171e^{−7.854t} − 1.171e^{−1.146t}
(c)
R(s)= 1/s
G(s)
9 / (s^2 + 2s + 9)
C(s)
Underdamped
jω
s-plane
j√8
−1
−j√8
σ
c(t) c(t) = 1−e^{−t}(cos√8t + √8/8 sin√8t)
=1−1.06e^{−t} cos(√8t−19.47°)
(d)
R(s)= 1/s
G(s)
9 / (s^2 + 9)
C(s)
Undamped
jω
s-plane
j3
−j3
σ
c(t)
c(t) = 1 − cos 3t
(e)
R(s)= 1/s
G(s)
9 / (s^2 + 6s + 9)
C(s)
Critically damped
jω
s-plane
−3
σ
c(t)
c(t) = 1 − 3te^{−3t} − e^{−3t}
t

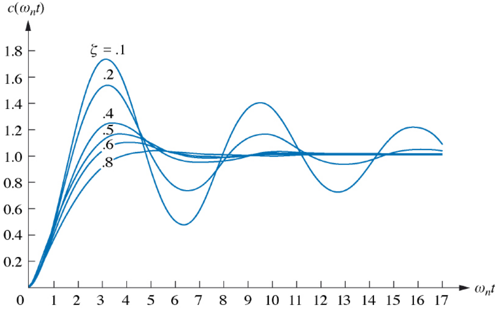


Category	Poles	$c(t)$
Overdamped	Two real: $-\sigma_1$, $-\sigma_2$	$K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$
Underdamped	Two complex: $-\sigma_d \pm j\omega_d$	$A e^{-\sigma_d t} \cos(\omega_d t - \phi)$
Undamped	Two imaginary: $\pm j\omega_n$	$A \cos(\omega_n t - \phi)$
Critically damped	Repeated real: $-\sigma_d$	$K_1 e^{-\sigma_d t} + K_2 t e^{-\sigma_d t}$

We can characterize the response of second-order systems using two parameters: ω_n and ζ

Natural Frequency, ω_n: This is the frequency of oscillation without damping. For example, the natural frequency of an RLC circuit with the resistor shorted, or of a mechanical system without dampers. An undamped system is described by its natural frequency.

Damping Ratio, ζ: This measures the amount of damping. For underdamped systems ζ lies in the range $[0, 1]$:



Damping ratio ζ is defined as follows:

$$\begin{aligned}\zeta &= \frac{\text{Exponential decay frequency}}{\text{Natural frequency}} \\ &= \frac{|\sigma_d|}{\omega_n}\end{aligned}$$

The exponential decay frequency σ_d is the real-axis component of the poles of a critically damped or underdamped system.

We now describe the general second-order system in terms of ω_n and ζ.

$$G(s) = \frac{b}{s^2 + as + b}$$

In other words we want to get the relationships from ω_n and ζ to a and b. Why? Because ω_n and ζ are more meaningful and useful for design.

If there were no damping, we would have a pure sinusoidal response. Thus, the poles would be on the imaginary axis and the TF would have the form,

$$G(s) = \frac{b}{s^2 + b}$$

The poles are at $s = \pm j\sqrt{b}$. The natural frequency is governed by the position of the poles on the imaginary axis. Therefore, $\omega_n = \sqrt{b}$.

$$b = \omega_n^2$$

Consider an underdamped system with poles $-\sigma_d \pm j\omega_d$. The exponential decay frequency is σ_d. For a general second-order system the denominator is $s^2 + as + b$ and the roots have real part $\sigma_d = -a/2$.

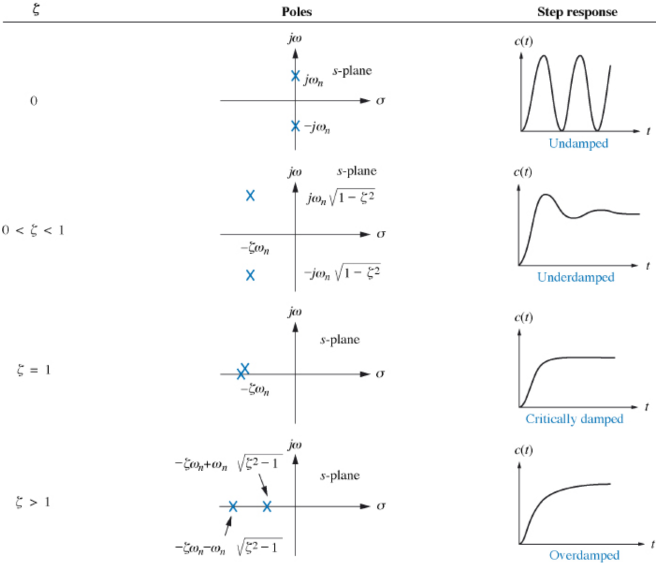
We apply the definition for ζ:

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency}} = \frac{|\sigma_d|}{\omega_n} = \frac{a/2}{\omega_n}$$

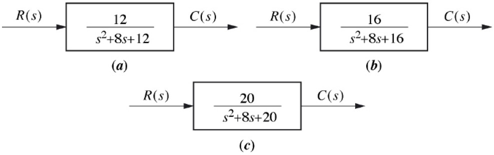
Thus, $a = 2\zeta\omega_n$. We can now describe the second-order system as follows:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Poles: $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$


ζ
Poles
Step response
0
jω
s-plane
$j\omega_n$
$-j\omega_n$
σ
c(t)
t
Undamped
0 < ζ < 1
$j\omega_n\sqrt{1-\zeta^2}$
$-\zeta\omega_n$
$-j\omega_n\sqrt{1-\zeta^2}$
Underdamped
ζ = 1
Critically damped
ζ > 1
$-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}$
$-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}$
Overdamped

e.g. Describe the category of the following systems:



$\omega_n = \sqrt{b}$, $\zeta = \frac{a/2}{\omega_n} = \frac{a}{2\sqrt{b}}$

(a) $\zeta = 1.155 \implies$ Overdamped

(b) $\zeta = 1 \implies$ Critically damped

(c) $\zeta = 0.894 \implies$ Underdamped

Characteristics of Underdamped Systems

Underdamped systems are very common and we will focus in particular on designing compensators for underdamped systems later in the course. Consider the step response for a general second-order system:

$$\begin{aligned} C(s) &= \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \\ &= \frac{K_1}{s} + \frac{K_2 s + K_3 s}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \end{aligned}$$

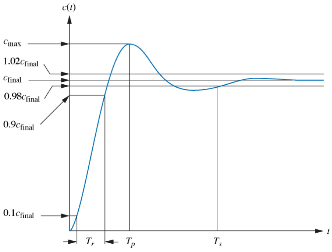
We solve for K_1, K_2, K_3 then take the ILT:

$$\begin{aligned} c(t) &= 1 - e^{-\zeta\omega_n t}\left[\cos(\omega_n\sqrt{1-\zeta^2}t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}t)\right] \\ &= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2 t} - \phi) \end{aligned}$$

where $\phi = \tan^{-1}\left(\zeta/\sqrt{1-\zeta^2}\right)$.

Although the two parameters ω_n and ζ completely characterize the form of the underdamped response, we usually specify the response with the following derived parameters:

- Peak time, T_p: The time required to reach the first (maximum) peak.
- Percent overshoot, $\%OS$: The amount that the response exceeds the final value at T_p.
- Settling time, T_s: The time required for the oscillations to die down and stay within 2% of the final value.
- Rise time, T_r: The time to go from 10% to 90% of the final value.



Consider determining T_p, the time required to reach the first peak. At the peak, the derivative is zero. Thus, we can solve for the value of t for which $\dot{c}(t) = 0$. We do this differentiation in the FD:

$$\begin{aligned} C(s) &= \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \\ \frac{d}{dt}c(t) &\rightarrow sC(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \end{aligned}$$

We now find the ILT to obtain $\dot{c}(t)$ and proceed to find the times at which $\dot{c}(t) = 0$.

COVERED ON BOARD

$$T_p = \frac{\pi}{\omega_n\sqrt{1 - \zeta^2}}$$

Percent overshoot is defined as follows,

$$\%OS = \frac{c_{max} - c_{final}}{c_{final}} \times 100$$

If the input is a unit step, $c_{final} = 1$.

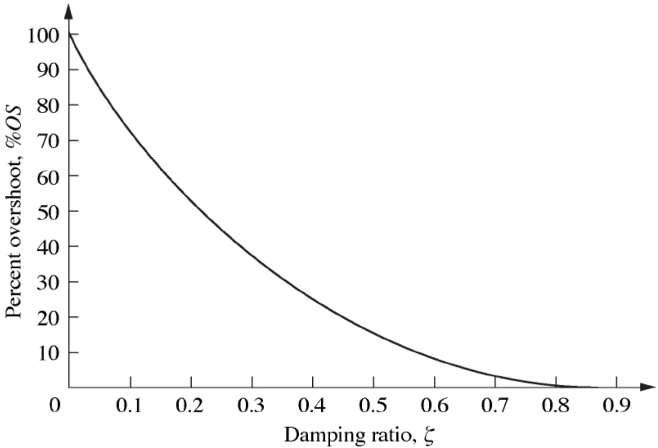
$$\begin{aligned} c(t) &= 1 - e^{-\zeta\omega_n t}\left[\cos(\omega_n\sqrt{1-\zeta^2}t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}t)\right] \\ c_{max} &= c(T_p) = 1 + e^{(-\zeta\pi/\sqrt{1-\zeta^2})} \end{aligned}$$

We obtain,

$$\%OS = e^{(-\zeta\pi/\sqrt{1-\zeta^2})} \times 100$$

This relationship is invertible,

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$


100
90
80
70
60
50
40
30
20
10
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Percent overshoot, %OS
Damping ratio, ζ

The settling time T_s is the time required for $c(t)$ to reach and stay within 2% of the final value.

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_n \sqrt{1-\zeta^2 t} - \phi)$$

Consider just the exponential envelope of $c(t)$,

$$\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}$$

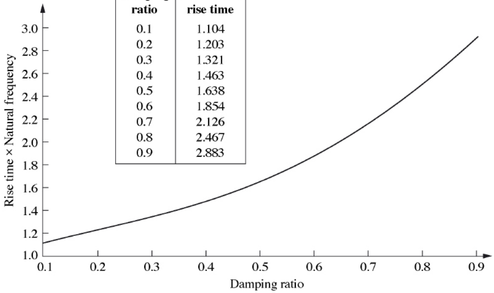
Solve for the time at which the envelope decays to 0.02

$$\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} = 0.02$$

$$T_s = \frac{-\ln(0.02\sqrt{1-\zeta^2})}{\zeta\omega_n} \approx \frac{4}{\zeta\omega_n}$$

Note that this is a conservative estimate since the sinusoid might actually reach and stay within 2% earlier.

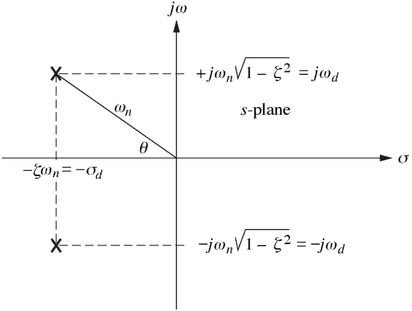
There is no analytical form for T_r (time to go from 10% to 90% of final value). This value can be calculated numerically and has been formed into a table:



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

Relationship to Pole Plot

The following is the pole plot for a general second-order system:



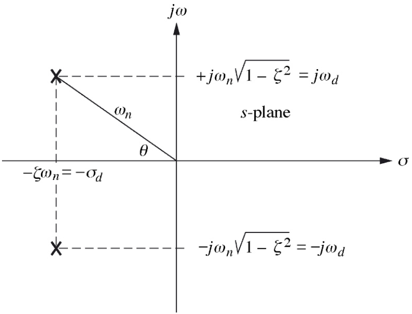
$\sigma_d = \zeta\omega_n$ is the real part of the pole and is called the *exponential decay frequency*.

$\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the imaginary part and is called the *damped frequency of oscillation*.

Notice the following:

- ω_n is the distance to the origin
- $\cos\theta = \zeta$

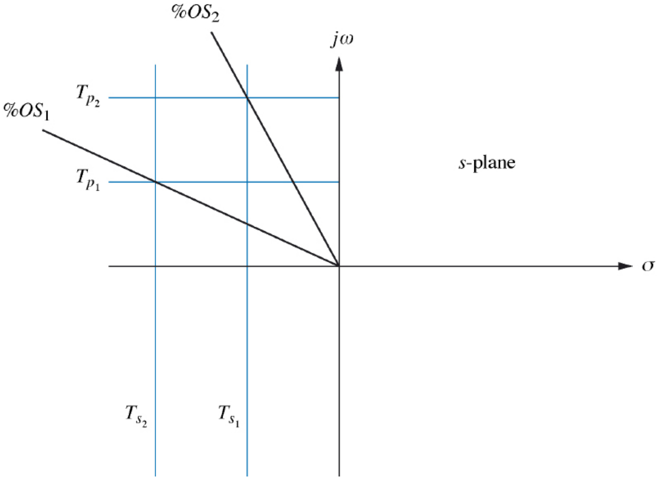
Relationship to Pole Plot

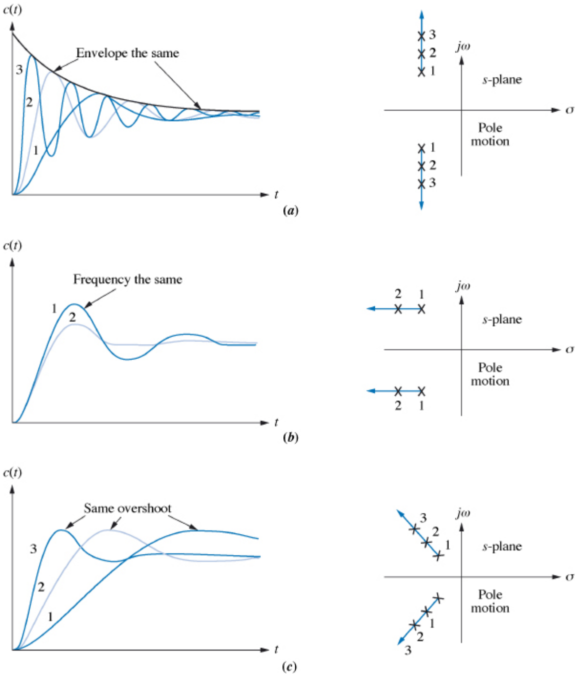


We can relate T_p, T_s, and $\%OS$ to the locations of the poles.

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} \qquad T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d} \qquad \%OS = f(\zeta)$$

$$T_p = \pi/\omega_d \qquad T_s = 4/\sigma_d$$

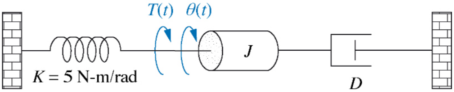




c(t)
Envelope the same
3
2
1
t
(a)
jω
s-plane
σ
Pole
motion
c(t)
Frequency the same
1
2
t
(b)
jω
s-plane
σ
Pole
motion
c(t)
Same overshoot
3
2
1
t
(c)
jω
s-plane
σ
Pole
motion

Design Example

Given the system below, find J and D to yield 20% overshoot and a settling time of 2 seconds for a step input torque $T(t)$.



The transfer function must first be determined,

$$G(s) = \frac{1/J}{s^2 + \frac{D}{J}s + \frac{K}{J}}$$

Relating to the standard form of a second-order systems we have,

$$\omega_n = \sqrt{\frac{K}{J}} \qquad 2\zeta\omega_n = \frac{D}{J}$$

The specification of 20% overshoot allows us to calculate $\zeta = 0.456$.

The specification of $T_s = 2$ allows us to calculate $\zeta\omega_n = 2$. From these values we can easily calculate $D = 1.04$ and $J = 0.26$.