# ENGI 7825: Linear Algebra Review
# Subpaces, Basis, Dimension, and Rank

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## Definition: Vector Space

A vector space is a set $\mathbb{V}$ which is closed under the following operations (where $\vec{x}, \vec{y}, \vec{z} \in \mathbb{V}$ and $c, t \in \mathbb{R}$):

- vector addition: $\vec{x} + \vec{y} \in \mathbb{V}$
- scalar multiplication: $c\vec{x} \in \mathbb{V}$

The following properties hold for any vector space:

1. $\vec{x} + \vec{y} \in \mathbb{V}$
2. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
3. $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
4. There is an element $\vec{0} \in \mathbb{V}$ so that $\vec{x} + \vec{0} = \vec{x} = \vec{0} + \vec{x}$
5. For each $\vec{x} \in \mathbb{V}$ there is a $\vec{-x} \in \mathbb{V}$ so that $\vec{x} + (\vec{-x}) = \vec{0}$
6. $c\vec{x} \in \mathbb{V}$
7. $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
8. $(c + t)\vec{x} = c\vec{x} + t\vec{x}$
9. $c(t\vec{x}) = (ct)\vec{x}$
10. $1\vec{x} = \vec{x}$

Interestingly, the concept of vector spaces applies to non-vectors. Anything that has definitions for addition and scalar multiplication that satisfy the closure property of vector spaces qualifies.

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Example: Let $P_n$ be all polynomials of degree at most $n$. (These can be viewed as functions $\mathbb{R} \to \mathbb{R}$.)

Specific Example: $P_5$ is every function we can write as

$$f(t) = a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t^1 + a_0$$

Can add any two of these or multiply by a scalar and you still get a member of $P_5$.

Example: if

$$f(t) = 3t^5 - 2t^4 + t^3 + 4t^2 + t^1 - 3$$

then

$$5f(t) = 15t^5 - 10t^4 + 5t^3 + 20t^2 + 5t^1 - 15$$

## Definition: Subspace

Suppose that $\mathbb{V}$ is a vector space and $\mathbb{U} \subset \mathbb{V}$. That is, $\mathbb{U}$ is contained in $\mathbb{V}$. Suppose further

- $\vec{0} \in \mathbb{U}$,
- for all $\vec{x}, \vec{y} \in \mathbb{U}$, the sum $\vec{x} + \vec{y} \in \mathbb{U}$
- for all $c \in \mathbb{R}$ and $\vec{x} \in \mathbb{U}$, the scalar product $c\vec{x} \in \mathbb{U}$

Then $\mathbb{U}$ is a subspace of $\mathbb{V}$

## Theorem

*Any subspace of a vector space is itself a vector space.*

Example: Let $P_5$ be the vector space of all $5^{th}$ degree polynomials. $P_4$ is a subspace of $P_5$.

non-Example Let $\mathbb{U} \subset \mathbb{R}^2$ be the set of all $\begin{bmatrix} x \\ y \end{bmatrix}$ so that $x \geq 0, y \geq 0$. Check if it's a subspace of $\mathbb{R}^2$:

- $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{U}$ ☑
- Suppose $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{U}$ and $\begin{bmatrix} x' \\ y' \end{bmatrix} \in \mathbb{U}$. Then $x, x' \geq 0 \implies x + x' \geq 0$ and $y, y' \geq 0 \implies y + y' \geq 0$. Hence $\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + x' \\ y + y' \end{bmatrix} \in \mathbb{U}$ ☑
- But, sadly, although $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{U}$, the scalar product $-1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \notin \mathbb{U}$. ☹
  Since we have shown some example does not satisfy one of the criteria, $\mathbb{U}$ is not a subspace.

Suppose $\mathbb{V}$ is any vector space and $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in \mathbb{V}$.

**Definition: Span (we have seen this before)**

Span$\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is the set of all elements of $\mathbb{V}$ that can be written as a linear combination of $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$.

Example: In the vector space $\mathbb{R}^3$ Span$\{[1\ 0\ 0]^T, [0\ 1\ 0]^T\}$ is the set of all vectors in the $xy-$plane, which is a subspace of $\mathbb{R}^3$.

**Theorem**

Span$\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ *is always a subspace of* $\mathbb{V}$

We can view the multiplication of a $m \times n$ matrix $A$ by a $n \times 1$ vector $\vec{x}$ in two distinct ways.

Dot product with rows: Each row of $\vec{y} = A\vec{x}$ consists of the dot product of the corresponding row of $A$ with $\vec{x}$. Let the $m$ rows of $A$ be the $\vec{r}$ vectors below.

$$A = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \dots \\ \vec{r}_m \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} \vec{r}_1\vec{x} \\ \vec{r}_2\vec{x} \\ \dots \\ \vec{r}_m\vec{x} \end{bmatrix}$$

Combination of the columns: The resultant vector $\vec{y} = A\vec{x}$ consists of the combination of columns of $A$ as given by the elements of $\vec{x}$. Let the $n$ columns of $A$ be the $\vec{c}$ vectors below.

$$A = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_n \end{bmatrix}$$

$$\vec{y} = \vec{c}_1 x_1 + \vec{c}_2 x_2 + \dots \vec{c}_n x_n$$

Let $A$ be an $m \times n$ matrix.

Consider all $\vec{u} \in \mathbb{R}^n$ so that $A\vec{u} = \vec{0}$. Call it the null space of $A$ or $\operatorname{Nul} A$.

We now show that $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^n$.

Need to check three things:

- $\vec{0} \in \operatorname{Nul} A$ (since $A\vec{0} = \vec{0}$).
- For any $\vec{x}$ & $\vec{y} \in \operatorname{Nul} A$,

  $$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}.$$

  Hence $\vec{x} + \vec{y} \in \operatorname{Nul} A$.
- Similarly, for any $c \in \mathbb{R}$ and $\vec{x} \in \operatorname{Nul} A$,

  $$A(c\vec{x}) = cA\vec{x} = c\vec{0} = \vec{0}.$$

  Hence $c\vec{x} \in \operatorname{Nul} A$.

Easy question: Given $\vec{u} \in \mathbb{R}^n$, is $\vec{u} \in \operatorname{Nul} A$?

Harder question: Describe all of $\operatorname{Nul} A$.

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Example: Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \qquad \vec{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Is $\vec{u}$ in $\operatorname{Nul} A$?

Answer: Just calculate $A\vec{u}$: $A\vec{u} = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \vec{u} \in \operatorname{Nul} A$.

Here

$$A\vec{u} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 - 2 \cdot 2 + 3 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 1 - 2 \cdot 3 + 5 \cdot 1 - 2 \cdot 0 \\ 1 \cdot 3 - 2 \cdot 8 + 13 \cdot 1 - 3 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence $\vec{u} \in \text{Nul}\, A$.

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Harder question: Describe all of Nul $A$. Interpretation: Find $\vec{u}_1, ...\vec{u}_k \in \text{Nul}\, A$ so that $\text{Span}\{\vec{u}_1, ...\vec{u}_k\} = \text{Nul}\, A$

Fancy description of an old problem we know how to do:

Solve the homogeneous system of linear equations given by $A\vec{x} = \vec{0}$.

Step 1: Convert $A$ to reduced echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \rightarrow_{1^{st}\ col.} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & 4 & -6 \end{bmatrix} \rightarrow_{2^{nd}\ col.} \begin{bmatrix} 1 & 0 & -1 & 7 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 2: Identify the free variables. Here they are $x_3, x_4$.

Step 3: Convert to vector equation:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - 7x_4 \\ -2x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

Then $\text{Nul}\, A = \text{Span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$

This spanning set is efficient: no proper subset spans $\text{Nul}\, A$.

As before, let $A$ be any $m \times n$ matrix:

Consider all linear combinations of the column vectors of $A$. It's a subset of $\mathbb{R}^m$, denoted Col $A$.

In other words, for

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$$

Col $A$ is Span$\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$.

Connection to $\vec{y} = A\vec{x}$: This system is solvable if and only if $\vec{y}$ is in the column space of $A$

$$\vec{y} \in \text{Col } A \iff \vec{y} = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots x_n\vec{a}_n \iff \vec{y} = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots x_n \end{bmatrix}$$

Describe Col $A$:

Answer: We've just done it – it's $\text{Span}\{\vec{a}_1, ..., \vec{a}_n\}$, the span of the column vectors of $A$.

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \implies \text{Col}\, A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 13 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} \right\}$$

But this is not necessarily an efficient description: a subset may span Col $A$. In this case, the first two vectors suffice:

$$\text{Col}\, A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix} \right\}$$

Harder question: Given $\vec{u} \in \mathbb{R}^m$, is $\vec{u} \in \text{Col}\, A$.

Translation: Are there $\{x_1, \ldots x_n\}$ so that

$$\vec{u} = x_1\vec{a}_1 + \ldots + x_n\vec{a}_n?$$

Equivalently:

Is there a vector $\vec{x} \in \mathbb{R}^n$ so that $A\vec{x} = \vec{u}$?

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Solution: Reduce augmented matrix $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n & \vec{u} \end{bmatrix}$ to echelon form and see if the equations are consistent:

Example: Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \qquad \vec{u} = \begin{bmatrix} 3 \\ 4 \\ 11 \end{bmatrix}$$

Is $\vec{u} \in \text{Col}\, A$?

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$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 3 \\ 1 & 3 & 5 & -2 & 4 \\ 3 & 8 & 13 & -3 & 11 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 2 & -3 & 1 \\ 0 & 2 & 4 & -6 & 2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last column says that $0x_1 + 0x_2 + 0x_3 + 0x_4 = 0u_3$ which is consistent. However, if the bottom-right entry had been non-zero then we would have had an inconsistent equation. So in this case $\vec{u} \in \text{Col}\, A$.

Additional payoff: First two columns are the pivot columns $\implies$ first two of the original columns span Col $A$.

## Definition

A subset $\{\vec{v}_1, \ldots \vec{v}_k\} \subset \mathbb{V}$ is a basis for $V$ if and only if it is linearly independent and it spans $V$.

Classic example: The set $\{\vec{e}_1, \ldots \vec{e}_n\} \subset \mathbb{R}^n$ is a basis for the vector space $\mathbb{R}^n$. Where

$$\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where the 1 appears in the $i^{th}$ position. This set is called the standard basis for $\mathbb{R}^n$.

**Theorem**

*Any finite spanning set of vectors contains a basis.*

Suppose $\{\vec{v}_1, \ldots \vec{v}_n\} \subset \mathbb{V}$ spans $\mathbb{V}$

- if $\{\vec{v}_1, \ldots \vec{v}_n\}$ is linearly independent, then it's a basis.
- if it's linearly dependent, some proper subset spans.

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Two important properties of a basis:

- No proper subset of a basis is a basis (it will no longer span).
- Adding an additional vector to a basis will no longer constitute a basis (no longer linearly independent).

A basis efficiently captures most information about a vector space.

Example:

Consider the set of 2-vectors

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$$

Is this set a basis for $\mathbb{R}^2$?

No. This set is not independent. But any pair of vectors from $S$ would form a basis for $\mathbb{R}^2$.

The dimension of a subspace is the number of vectors needed to form a basis.

Its clear that the dimension of $\mathbb{R}^n$ is $n$. Any vector in $\mathbb{R}^n$ can be written as a combination of $n$ basis vectors. The standard basis for $\mathbb{R}^n$ is the set $\{e_1, e_2, \ldots, e_n\}$. Other basis vectors are possible, but a potential set must contain $n$ independent vectors in order to qualify as a basis.

As we have seen, for a matrix $A$ there are two particularly interesting subspaces:

- The null space, $\operatorname{Nul} A$, consisting of all solutions to $A\vec{x} = 0$
- The column space, $\operatorname{Col} A$, consisting of all linear combinations of the columns of $A$. If there is a solution to $A\vec{x} = \vec{y}$ then $\vec{y}$ must lie in the column space of $A$ (i.e. it must be some linear combination of the columns of $A$).

The dimension of $\operatorname{Nul} A$ is known as nullity($A$).

The dimension of $\operatorname{Col} A$ is $rank(A)$. In fact, the rank is also the dimension of the row space.

**Sylvester's Law of Nullity**

For the *mxn* matrix $A$:

$$rank(A) + nullity(A) = n$$

Example: Given a matrix $A$ find a basis for its null space and the dimension of that basis.

Answer: We've done this! Nul $A$, can be found via reduced echelon form.

Previous example: Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 7 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Nul $A$ spanned by one vector for each free variable:

Here the free variables are $x_3, x_4$. First equation gives:

$$x_1 - x_3 + 7x_4 = 0 \implies x_1 = x_3 - 7x_4$$

Second equation gives:

$$x_2 + 2x_3 - 3x_4 = 0 \implies x_2 = -2x_3 + 3x_4$$

Combining: $\vec{x} \in \operatorname{Nul} A \iff$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - 7x_4 \\ -2x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

Hence $\operatorname{Nul} A = \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$

The bottom rows show that these vectors are linearly independent, so they are a basis for $\operatorname{Nul} A$. Since there are two, $\text{nullity}(A) = 2$.

Example: Given a matrix $A$ find a basis for its column space and its dimension.

Step 1: Reduce to echelon form.

Step 2: Identify the pivot columns

Step 3: These columns of the original matrix! are a basis for Col $A$.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \rightarrow_{row\ reduce} \begin{bmatrix} 1 & 0 & -1 & 7 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns are first and second columns

First and second columns of original $A$ span Col $A$:

Col $A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix} \right\}$ These vectors form a basis for Col $A$.

Note that Sylvester's Law of Nullity is satisfied:

$rank(A) + nullity(A) = n$