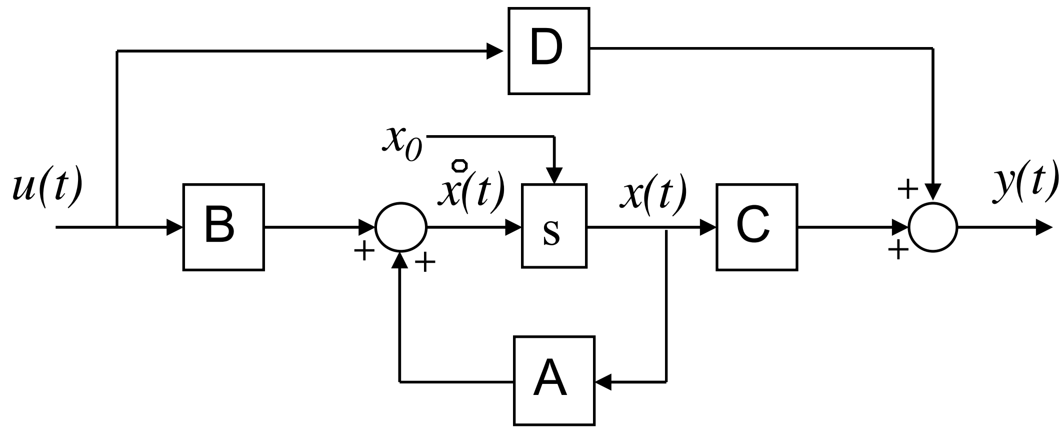


ENGI 7825: Control Systems II

State Feedback: Part 2

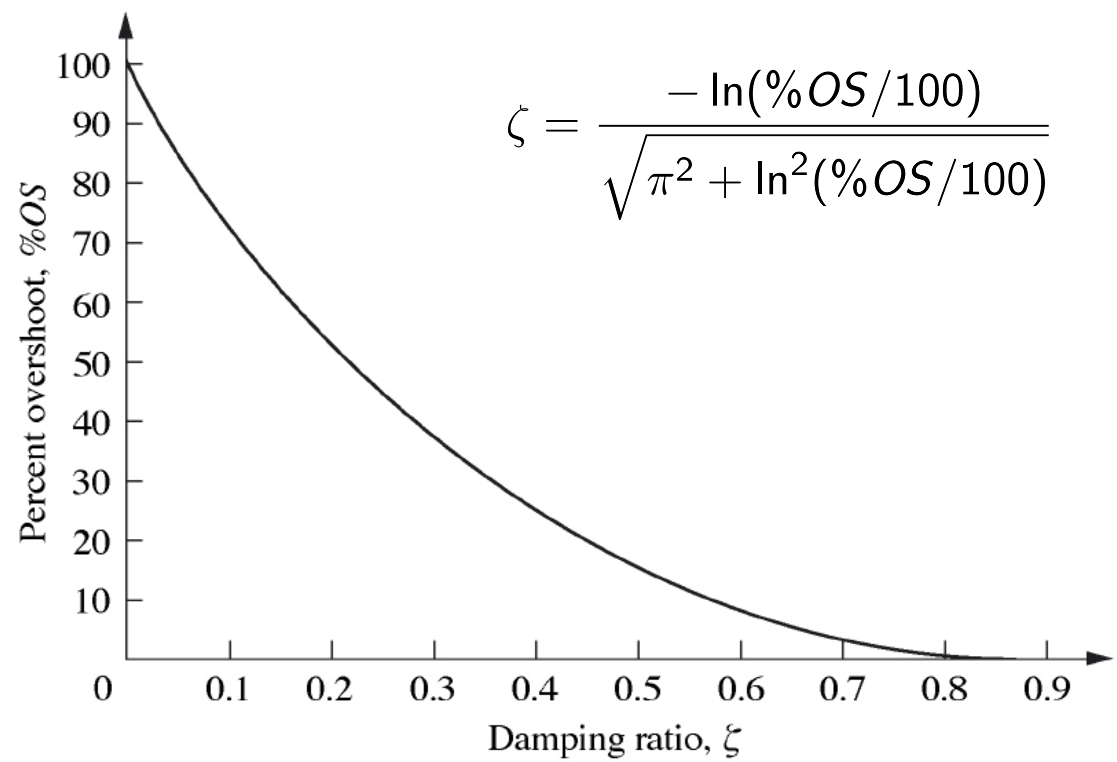
Instructor: Dr. Andrew Vardy

Adapted from the notes of
Gabriel Oliver Codina

2^{nd} Order Design Constraints

- Specifications such as T_s, T_p, and %OS are usually specified as inequalities. For example,
  - %OS <= 4%
  - T_s <= 2 s
  - T_p <= 0.5 s

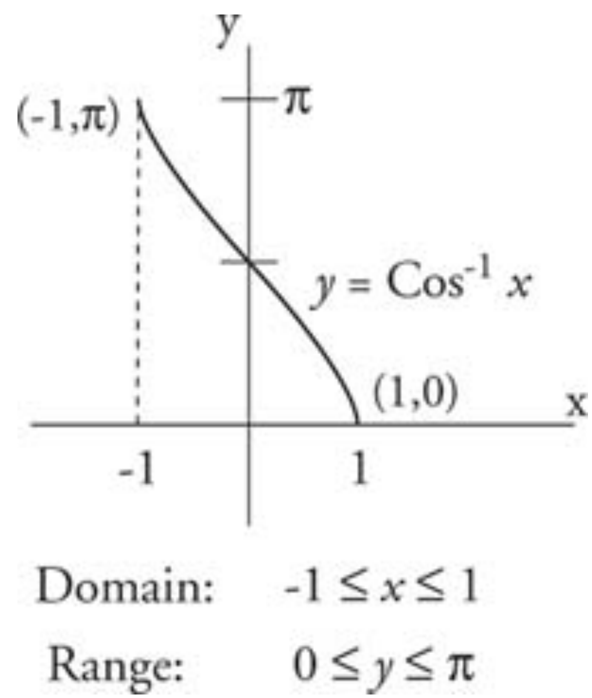
- Consider %OS
  - An upper bound on %OS is a lower bound on ζ.
  - For %OS <= 4% we get ζ >= 0.716



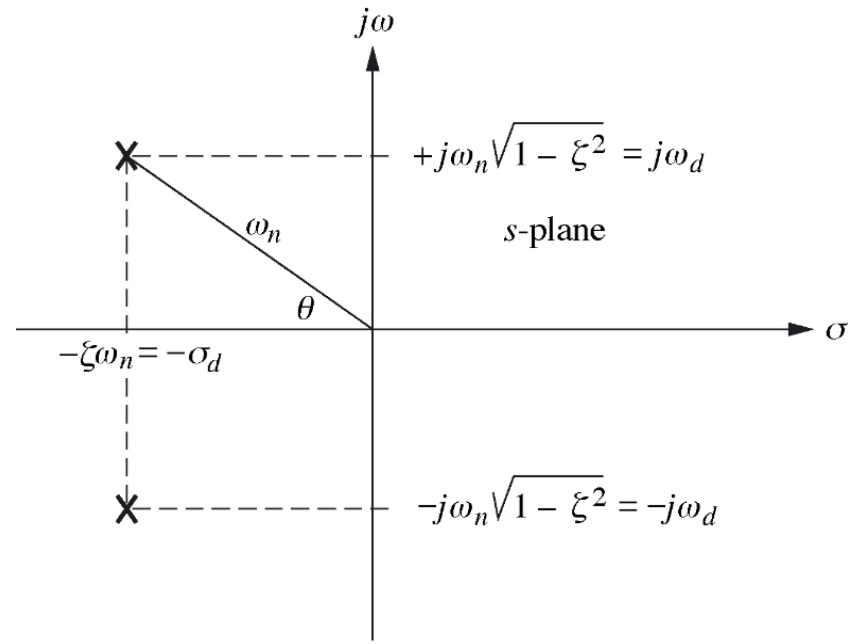
$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

$\zeta >= 0.716$

- With θ defined as the angle of the complex poles to the negative real axis, we can utilize $\theta = \cos^{-1}(\zeta) = 44.3^o$.
- Since $\cos^{-1}(\zeta)$ is a decreasing function of ζ, the lower bound on zeta becomes an upper bound on θ: $\theta <= 44.3^o$.
  - Exercise: draw the regions of acceptable eigenvalues for %OS



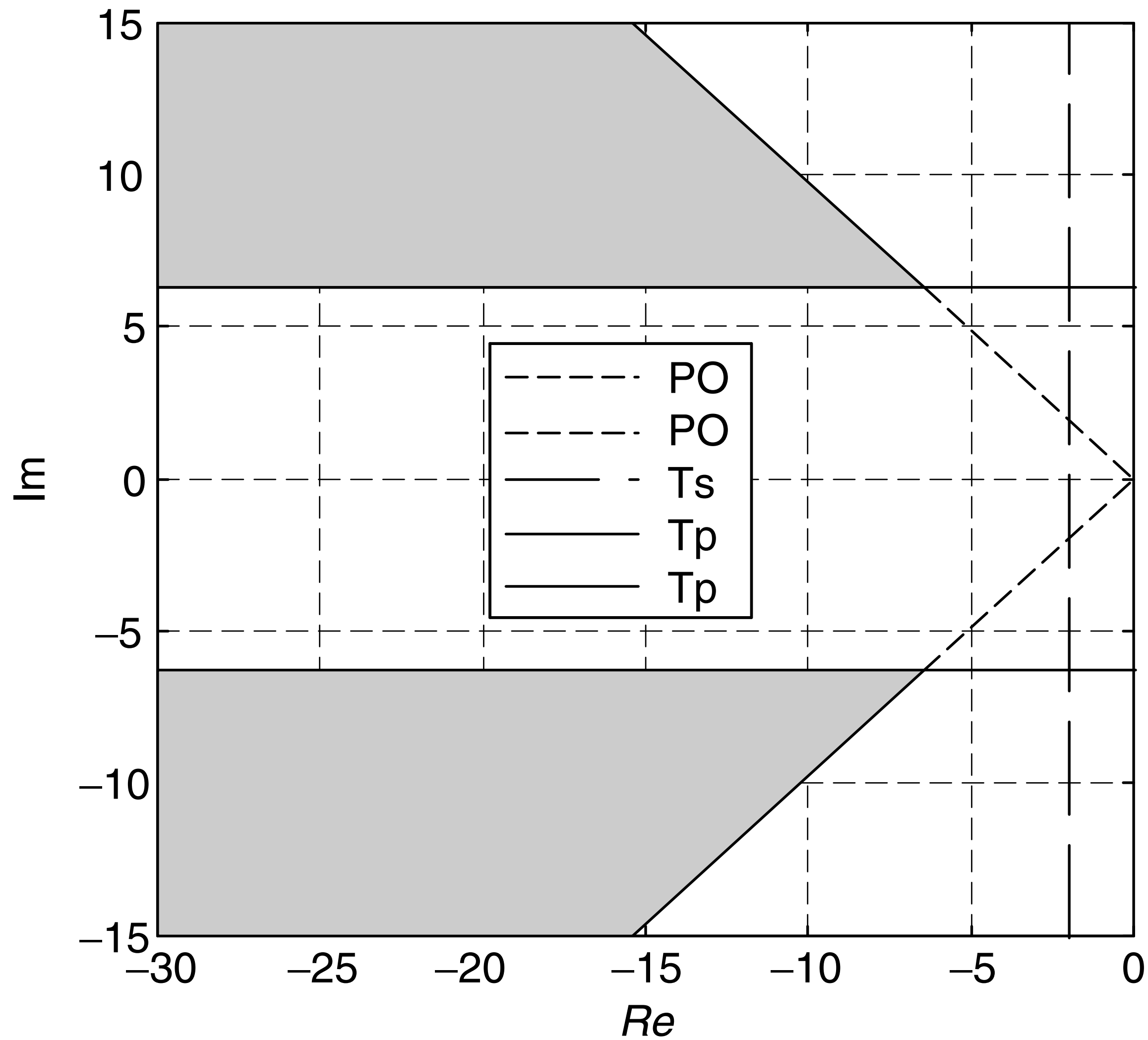
- Now consider the specification $T_s <= 2$ s.
- The real-part of the eigenvalue pair should lie to the left of $\sigma_d >= 4 / 2 = 2$
  - That is, $-\sigma_d <= -2$



- Finally we consider $T_p <= 0.5$ s
- Using the formula: $\omega_d >= \pi / 0.5 = 2\pi$
- This means that the absolute value of the imaginary-part of the eigenvalue pair must lie above 2π

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d}$$

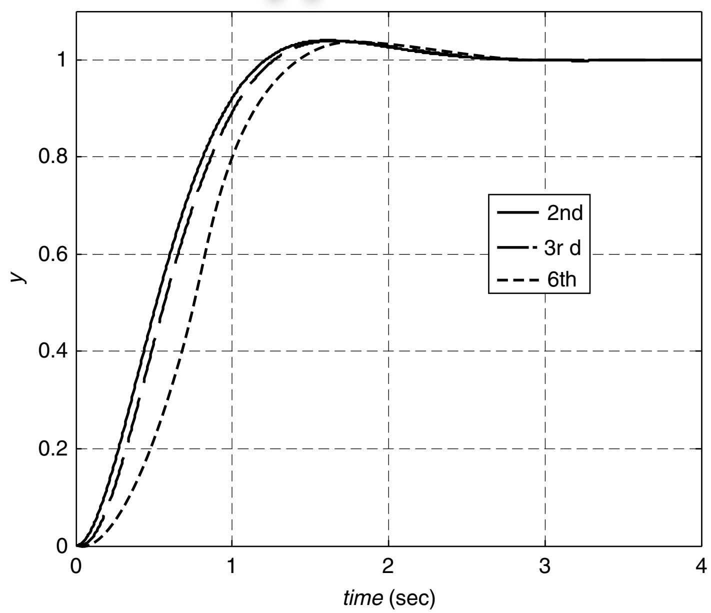
$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$



The shaded regions show acceptable eigenvalue locations. Note that the constraint that $T_s <= 2$ s (which led to $-\sigma_d <= -2$) is made redundant by the other two constraints.

Higher-Order Systems

- The behaviour of higher-order systems is dictated by the pair of poles that lie closest to the origin
  - These are the dominant 2nd order poles
- If the other poles lie significantly to the left, then they lead to quickly decaying responses
  - See supplementary notes from 5821: “Systems with Additional Poles or Zeros”
- The following are 3rd and 6th order systems that are 2nd order in appearance:

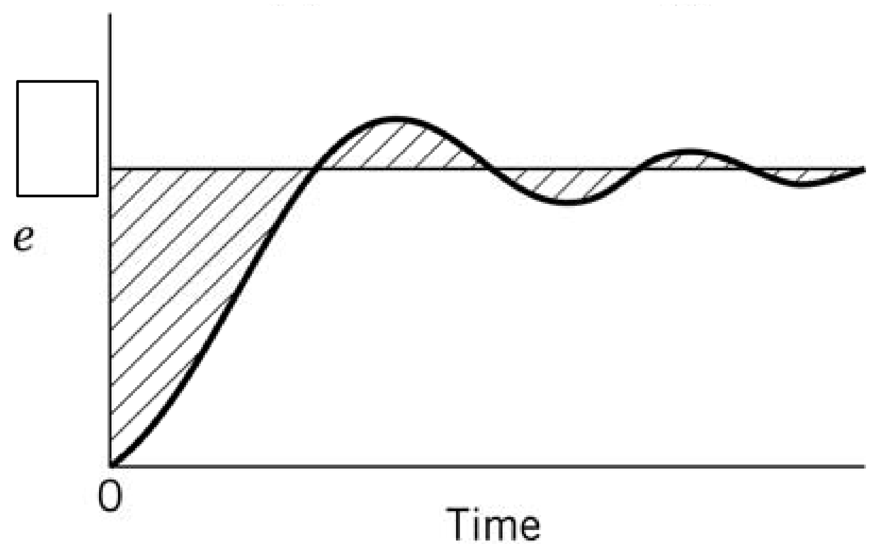


System Order	Eigenvalues
Second	$\lambda_{1,2} = -2 \pm 1.95i$
Third	$\lambda_{1,2,3} = -2 \pm 1.95i, -20$
Sixth	$\lambda_{1,2,3,4,5,6} = -2 \pm 1.95i, -20, -21, -22, -23$

Systems which Minimize ITAE

- If we have full control of our system's characteristic polynomial, why not place the eigenvalues in the "best" possible positions.
- How can "best" be defined? One possible definition for a step response is the one which minimizes ITAE (integral of time multiplied by absolute error):

$$\text{ITAE} = \int_0^\infty t|e(t)|dt$$



- The shaded areas indicate |e(t)|
- We multiply by time to further penalize error that occurs later in the response

- The following polynomials are those which minimize ITAE:

System Order	Characteristic Polynomial
First	$s + \omega_n$
Second	$s^2 + 1.4\ \omega_n s + \omega_n^2$
Third	$s^3 + 1.75\ \omega_n s^2 + 2.15\ \omega_n^2 s + \omega_n^3$
Fourth	$s^4 + 2.1\ \omega_n s^3 + 3.4\ \omega_n^2 s^2 + 2.7\ \omega_n^3 s + \omega_n^4$
Fifth	$s^5 + 2.8\ \omega_n s^4 + 5.0\ \omega_n^2 s^3 + 5.5\ \omega_n^3 s^2 + 3.4\ \omega_n^4 s + \omega_n^5$
Sixth	$s^6 + 3.25\ \omega_n s^5 + 6.6\ \omega_n^2 s^4 + 8.6\ \omega_n^3 s^3 + 7.45\ \omega_n^4 s^2 + 3.95\ \omega_n^5 s + \omega_n^6$

- To obtain the full transfer function we must choose a value for !$_n$ and then utilize the following (where $d_k(s)$ is the k^{th} order polynomial from the table above):

$$H_k(s) = \frac{\omega_n^k}{d_k(s)}$$

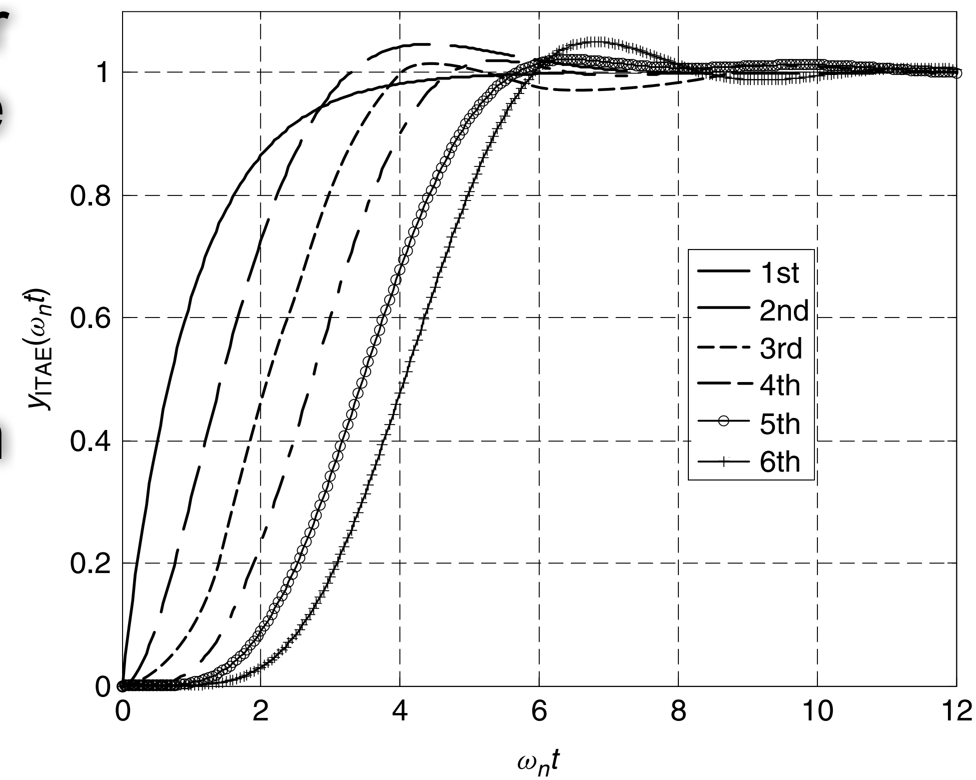


FIGURE 7.8 ITAE unit step responses.

Example 2

- We are given the following system:

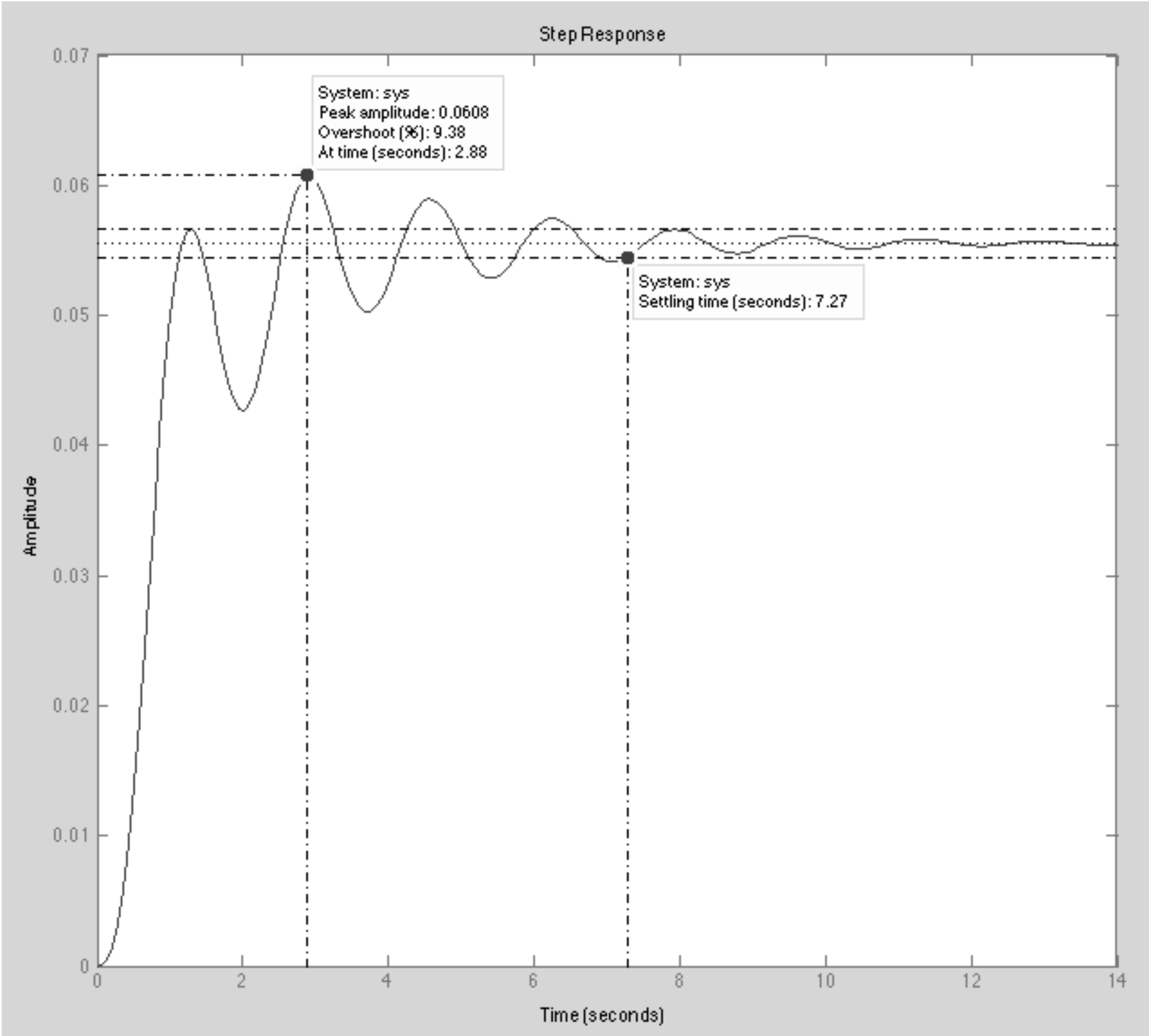
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -18 & -15 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

- By inspection we can obtain the open-loop characteristic polynomial and its eigenvalues:

$$a(s) = s^3 + a_2 s^2 + a_1 s + a_0 = s^3 + 2s^2 + 15s + 18$$

$$\lambda_{1,2,3} = -1.28, -0.36 \pm j3.73$$

- The step response is ...


Step Response
System: sys
Peak amplitude: 0.0608
Overshoot (%): 9.38
At time (seconds): 2.88
System: sys
Settling time (seconds): 7.27
Amplitude
Time (seconds)
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
2
4
6
8
10
12
14

$$a(s) = s^3 + a_2 s^2 + a_1 s + a_0 = s^3 + 2s^2 + 15s + 18$$

- We have the following desired characteristics:
  - T_s = 3 seconds
  - % OS = 6%
  - Maintain steady-state value of the original system
- We will design a third-order characteristic polynomial that consists of two dominant 2nd order poles and a third pole, lying further to the left
- Under the assumption that our system is purely 2nd order we arrive at the following parameters:
  - ζ >= 0.67 (we will choose ζ = 0.67)
  - ω_n >= 2 rad / s (we will choose ω_n = 2)
  - Eigenvalues: $\lambda_1, \lambda_2 = -1.33 \pm j\,1.49$
- We place the additional eigenvalue 10 units to the left and on the real-axis: $\lambda_3 = -13.33$
- This gives us the following desired characteristic polynomial and K matrix:

$$\alpha(s) = (s + 1.33 + j1.49)(s + 1.33 - j1.49)(s + 13.33)$$
$$= s^3 + 16s^2 + 39.55s + 53.26$$

$$K_{\mathrm{CCF}} = \begin{bmatrix} (\alpha_0 - a_0) & (\alpha_1 - a_1) & (\alpha_2 - a_2) \end{bmatrix}$$
$$= \begin{bmatrix} (53.26 - 18) & (39.55 - 15) & (16 - 2) \end{bmatrix}$$
$$= \begin{bmatrix} 35.26 & 24.55 & 14.00 \end{bmatrix}$$

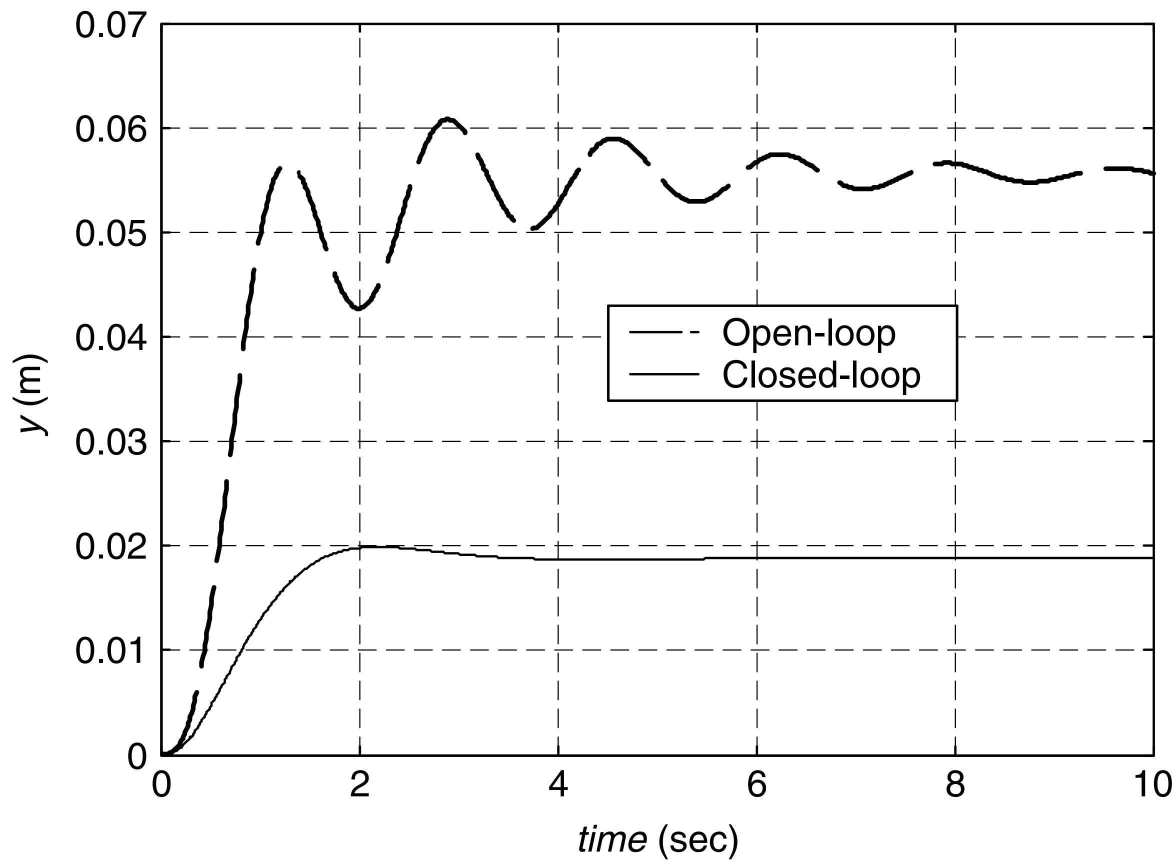
Using $u(t) = -K_{\text{CCF}}\ x_{\text{CCF}}(t) + r(t)$ (that is, G = 1)

we get the following compensated system:

$$A_{\text{CCF}} - B_{\text{CCF}}K_{\text{CCF}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -53.26 & -39.55 & -16 \end{bmatrix} \quad B_{\text{CCF}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C_{\text{CCF}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

The transient response looks much nicer (and has the desired characteristics) but the goal was to maintain the steady-state value of 0.056.



$$H_{open}(s) = \frac{1}{s^3 + 2s^2 + 15s + 18} \qquad H_{closed}(s) = \frac{1}{s^3 + 16s^2 + 39.55s + 53.26}$$

$$H_{open}(0) = \frac{1}{18} = 0.056 \qquad H_{closed}(0) = \frac{1}{53.26} = 0.0188$$

To maintain the desired steady-state value, we incorporate the gain:
G = 0.056 / 0.0188 = 2.96