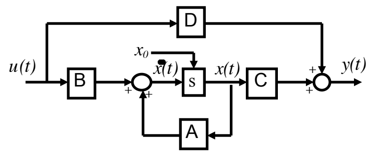


ENGI 7825: Control Systems II

State Feedback: Part 1

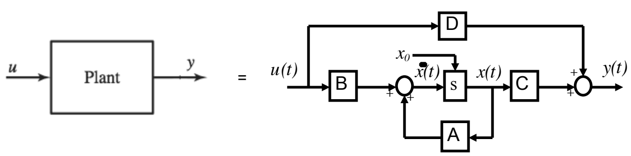
Instructor: Dr. Andrew Vardy

Adapted from the notes of
Gabriel Oliver Codina

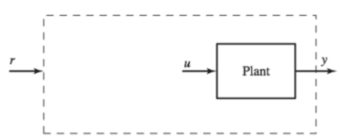
Introduction

- The objective is to design control laws that yield desirable closed-loop performance in terms of both transient and steady-state response characteristics.
- If the open-loop state equation is controllable, then an arbitrary closed-loop eigenvalue placement via state-space feedback can be achieved.
  - Various names for the same technique:
    - **State feedback**
    - Eigenvalue placement
    - Pole placement
- Assumptions:
  - The system must be controllable
  - We must have access to all state variables

- So far we have just considered the plant without any imposed control. The direct input to the plant is u(t):



- If the plant's open-loop response is unsatisfactory then incorporate a new input called r(t):



State Feedback Control Law

The open-loop system under study (*the plant*) is represented by the LTI state equation:

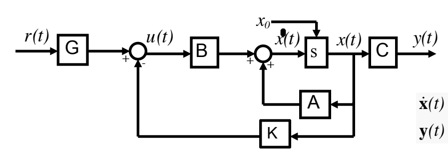
$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

(null direct matrix D is assumed)

We focus on the resulting effect of state feedback control laws like:

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) + \mathbf{G}\mathbf{r}(t)$$

where K is the constant state feedback gain matrix (m x n) that yields the closed-loop state equation with the desired performance characteristics and G is an *(m x p)* matrix which scales the new reference input r(t) so that the magnitude of y(t) matches r(t)



$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{G}\mathbf{r}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Open-loop system

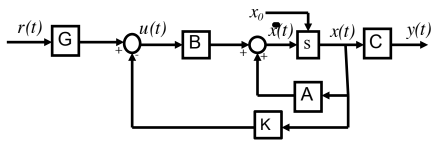
$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) + \mathbf{G}\mathbf{r}(t)$$

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{G}\mathbf{r}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Closed-loop system

- Why has u(t) been re-defined in this way?

r(t) is called the reference input. It is set to the desired value of y(t).

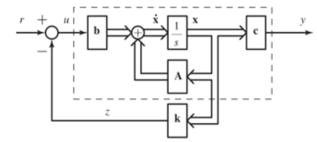


- u(t) now plays a similar role to an error signal in classical control. It gives the difference between the reference input r(t) and the system state, scaled through K.
  - However, the feedback now comes from x(t) not y(t).

Application to SISO Systems

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) + \mathbf{G}\mathbf{r}(t)$$

- If the system is SISO:
  - u(t) is a scalar
  - r(t) is a scalar
  - K is a 1 x n row vector
  - G is a scalar



- We can re-write $\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) + \mathbf{G}\mathbf{r}(t)$ as:

$$u(t) = -\begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + Gr(t) = -k_1 x_1(t) - k_2 x_2(t) - \cdots - k_n x_n(t) + Gr(t)$$

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{G}\mathbf{r}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

- **Feedback Gain Formula for Controller Canonical Form (CCF)**

The coefficient matrices for CCF are given below:

$$\mathbf{A}_{CCF} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix} \quad \mathbf{B}_{CCF} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

its characteristic polynomial (transfer function denominator) is written as:

$$s^n + a_{n-1}s^{n-1} + \cdots a_2 s^2 + a_1 s + a_0$$

For the single-input case, the gain matrix **K** is reduced to a feedback gain vector denoted by: $\mathbf{K}_{CCF} = [k_0 \ k_1 \ k_2 \cdots k_{n-1}]$

Thus, the closed-loop system dynamics matrix is:

$$\mathbf{A}_{CCF} - \mathbf{B}_{CCF}\mathbf{K}_{CCF} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 - k_0 & -a_1 - k_1 & -a_2 - k_2 & \ldots & -a_{n-1} - k_{n-1} \end{bmatrix}$$

with char. poly: $s^n + (a_{n-1} + k_{n-1})s^{n-1} + \cdots (a_2 + k_2)s^2 + (a_1 + k_1)s + (a_0 + k_0)$

Closed-loop Eigenvalue Placement

The characteristic polynomial for the compensated system:

$$s^n + (a_{n-1} + k_{n-1})s^{n-1} + \cdots (a_2 + k_2)s^2 + (a_1 + k_1)s + (a_0 + k_0)$$

The following represents the desired characteristic polynomial:

$$s^n + \alpha_{n-1}s^{n-1} + \cdots \alpha_2 s^2 + \alpha_1 s + \alpha_0$$

These alpha coefficients would be arrived at by looking at the constraints imposed by the problem (e.g. specifying a desired %OS and T_s).

We determine $\mathbf{K}_{CCF}$ by pairing, term-by-term, the two polynomials above (the actual and the desired) and solving for the k values.

$$\alpha_0 = a_0 + k_0 \quad \alpha_1 = a_1 + k_1 \quad \alpha_2 = a_2 + k_2 \ \cdots \ \alpha_{n-1} = a_{n-1} + k_{n-1}$$

which yields:

$$\mathbf{K}_{CCF} = \left[(\alpha_0 - a_0) \quad (\alpha_1 - a_1) \quad \cdots \quad (\alpha_{n-1} - a_{n-1})\right]$$

Dynamic Response Shaping

- We are interested in shaping the transient response (often the step response), by modifying such characteristics as rise time (Tr), peak time (Tp), percent overshoot (%OS), and settling time (Ts)
- Second-order dominant systems are frequently used as approximations in the design process
  - That is if the system is 3rd order or higher, it is approximated as 2nd order
  - If the system is 1st or 2nd order then it is treated as such
- Lets have a quick review of the characteristics of 1st and 2nd order systems:
  - [Time response notes from 5821]

Summary of 1st and 2nd Order System Characteristics

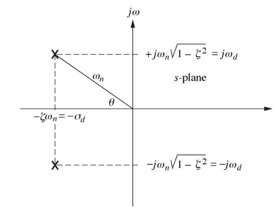
First-Order System

$$G(s) = \frac{a}{s+a} \qquad T_r = \frac{2.2}{a} \qquad T_s = \frac{4}{a}$$

Second-Order System

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

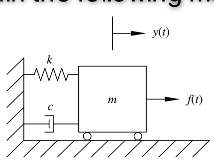
$$cos\theta = \zeta$$



$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} \qquad T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d} \qquad \%OS = e^{(-\zeta\pi/\sqrt{1-\zeta^2})} \times 100 \qquad \zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

Example 1

- Consider again the following mechanical system:



State variables:

$$x_1(t) = y(t)$$
$$x_2(t) = \dot{y}(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k}{m} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$H(s) = \frac{\frac{k}{m}}{s^2 + \frac{c}{m}s + \frac{k}{m}}$$

$$H(s) = \frac{\frac{k}{m}}{s^2 + \frac{c}{m}s + \frac{k}{m}} \qquad \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- We set specific values of the constants and compare with the standard 2nd order transfer function to obtain:

$$m = 1 \text{ kg}, \; c = 1 \text{ N-s/m}, \text{ and } k = 10 \text{ N/m}.$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10}{1}} = 3.16 \text{ rad/s}$$

$$\xi = \frac{c}{2\sqrt{km}} = \frac{1}{2\sqrt{10(1)}} = 0.158$$

Eigenvalues:

$$\lambda_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$$\lambda_{1,2} = -0.5 \pm 3.12i$$

Step Response:

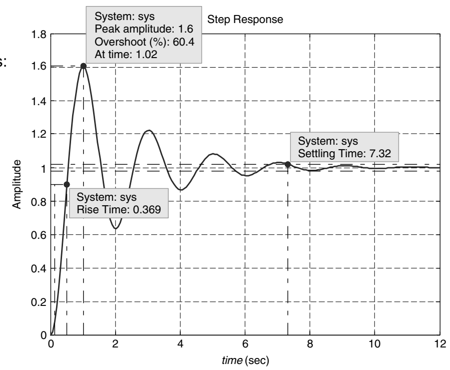
$$y(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)$$

$$y(t) = 1 - 1.01e^{-0.5t}\sin(3.12t + 80.9^\circ)$$

Calculated characteristics:

T_s = 8 seconds

%OS = 60.46%



Both of these characteristics can be reduced! Desired:

T_s = 2 seconds

%OS = 4 %

In Matlab:

- Generate LTI system **sys** using **tf** or **ss**
- Generate step response with **step(sys)**
- Right-click on plot and select “Characteristics” then the characteristic of interest
- Click on generated point to see the value selected.

- We have the following desired characteristics:
  - T_s = 2 seconds
  - % OS = 4%
- From these we can determine the desired 2[nd] order system parameters

$$\xi' = \frac{\left|\ln\left(\frac{PO}{100}\right)\right|}{\sqrt{\pi^2 + \left[\ln\left(\frac{PO}{100}\right)\right]^2}} = \frac{\left|\ln\left(\frac{4}{100}\right)\right|}{\sqrt{\pi^2 + \left[\ln\left(\frac{4}{100}\right)\right]^2}} = 0.716$$

$$\omega_n' = \frac{4}{\xi' t_S} = 2.79 \text{ rad/s}$$

- Desired characteristic polynomial:

$$\lambda^2 + 2\xi'\omega_n'\lambda + {\omega_n'}^2 = \lambda^2 + 4\lambda + 7.81$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Open-loop system

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) + \mathbf{G}\mathbf{r}(t)$$

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{G}\mathbf{r}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Closed-loop system

- Desired characteristic polynomial:

$$\lambda^2 + 2\xi'\omega_n'\lambda + {\omega_n'}^2 = \lambda^2 + 4\lambda + 7.81$$

- Open-loop (i.e. original) characteristic polynomial:

$$\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 = \lambda^2 + \lambda + 10$$

- A – BK matrix for the compensated system:

$$A - BK = \begin{bmatrix} 0 & 1 \\ -10 - k_0 & -1 - k_1 \end{bmatrix}$$

- Characteristic polynomial for A – BK:

$$\lambda^2 + (1 + k_1)\lambda + (10 + k_0)$$

- So we need k_1 = 3 and k_0 = -2.19

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Open-loop system

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) + \mathbf{G}\mathbf{r}(t)$$

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{G}\mathbf{r}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Closed-loop system

$$A = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ -10 - k_0 & -1 - k_1 \end{bmatrix}$$

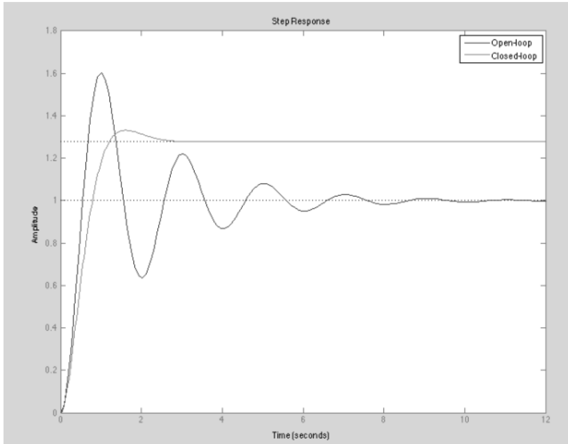
Are we done? Not yet. Assuming G = 1, what transfer functions would we obtain?

Obtain transfer functions from A, B, and C (open-loop) or A-BK, BG, and C (closed-loop):

$$H_{open}(s) = C(sI - A)^{-1}B + D = \frac{10}{s^2 + s + 10}$$

$$H_{closed}(s) = C(sI - A + BK)^{-1}BG + D = \frac{10}{s^2 + 4s + 7.81}$$

Not in pure 2[nd] order form so the step response will not go to 1



$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
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$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) + \mathbf{G}\mathbf{r}(t)$$

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{G}\mathbf{r}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Open-loop system

Closed-loop system

$$H_{closed}(s) = C(sI - A + BK)^{-1}BG + D = \frac{10}{s^2 + 4s + 7.81}$$

- Consider the step response for this system:

$$Y(s) = H(s)\frac{1}{s}$$

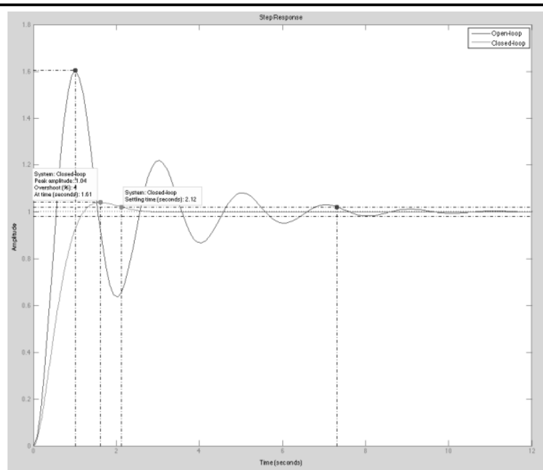
- What's the steady-state value? We can use the **final-value theorem** to find out:

$$\text{The Final Value Theorem: } f(\infty) = \lim_{s\to 0} sF(s)$$

$$y(\infty) = \lim_{s\to 0} H(s) = \frac{10}{7.81}$$ H(s=0) also known as **DC gain**

- If we want a steady-state value of y(1) = 1 then set G = 7.81 / 10

$$H(s) = \frac{7.81}{s^2 + 4s + 7.81}$$



Success! %OS = 4 as desired. However, T_s = 2.12 which is a little larger than 2. Recall that our formula for T_s is actually based on an approximation, so this is a good result