

ENGI 7825: Control Systems II

State Feedback: Part 1

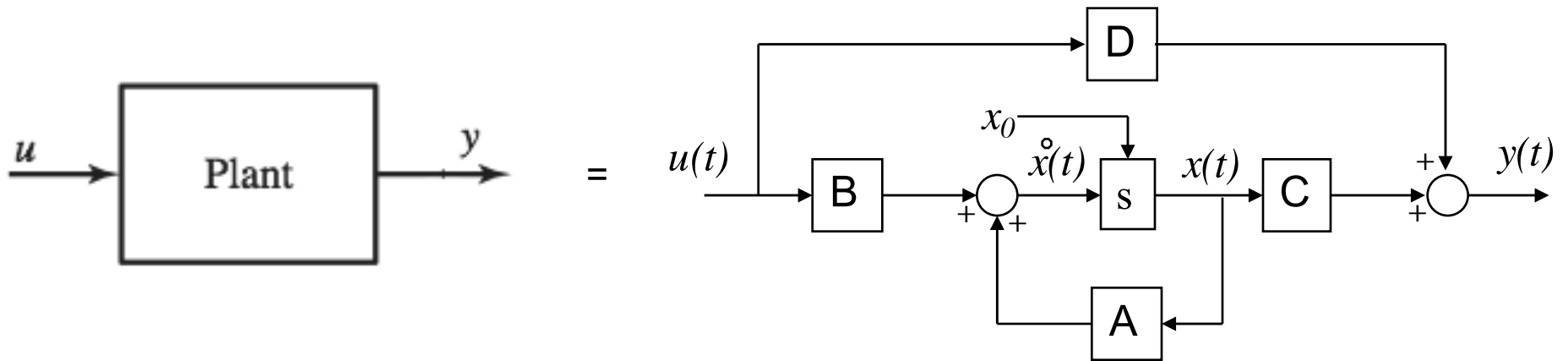
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Adapted from the notes of
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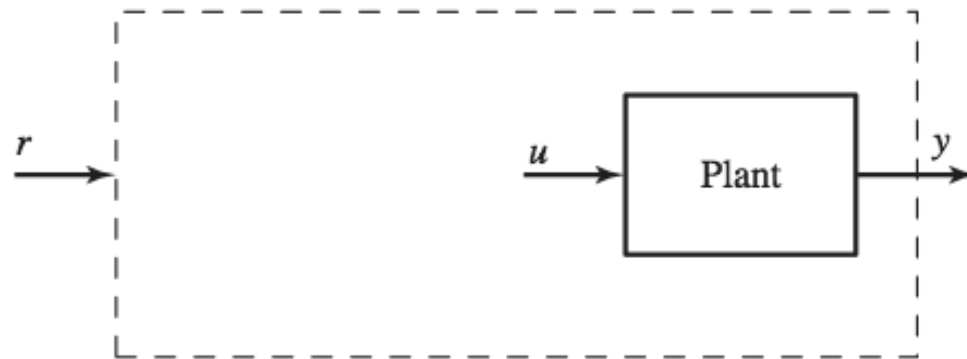
Introduction

- ▶ The objective is to design control laws that yield desirable closed-loop performance in terms of both transient and steady-state response characteristics.
- ▶ If the open-loop state equation is controllable, then an arbitrary closed-loop eigenvalue placement via state-space feedback can be achieved.
 - Various names for the same technique:
 - ▶ **State feedback**
 - ▶ Eigenvalue placement
 - ▶ Pole placement
- ▶ Assumptions:
 - The system must be controllable
 - We must have access to all state variables

- So far we have just considered the plant without any imposed control. The direct input to the plant is $u(t)$:



- If the plant's open-loop response is unsatisfactory then incorporate a new input called $r(t)$:



State Feedback Control Law

The open-loop system under study (*the plant*) is represented by the LTI state equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

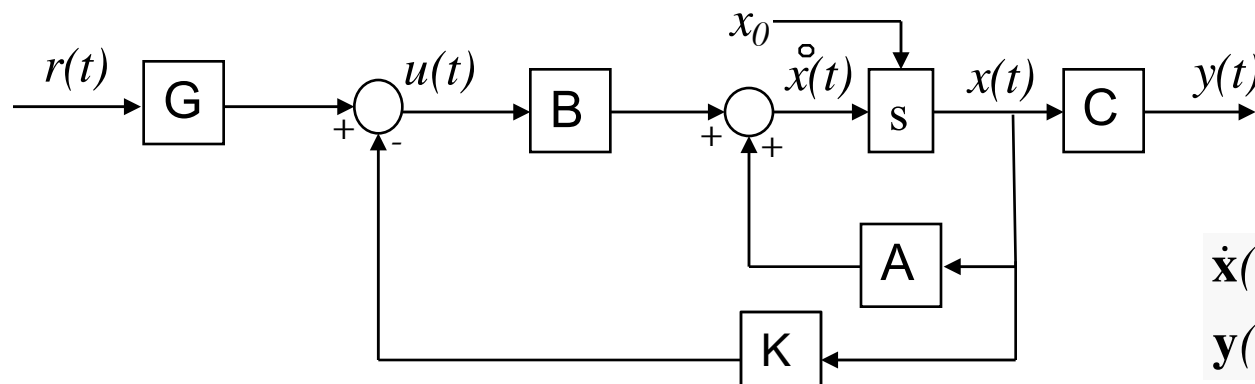
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

(null direct matrix \mathbf{D} is assumed)

We focus on the resulting effect of state feedback control laws like:

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) + \mathbf{G}\mathbf{r}(t)$$

where \mathbf{K} is the constant state feedback gain matrix ($m \times n$) that yields the closed-loop state equation with the desired performance characteristics and \mathbf{G} is an ($m \times p$) matrix which scales the new reference input $r(t)$ so that the magnitude of $y(t)$ matches $r(t)$



$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{BK})\mathbf{x}(t) + \mathbf{B}\mathbf{G}\mathbf{r}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Open-loop system

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) + \mathbf{G}\mathbf{r}(t)$$

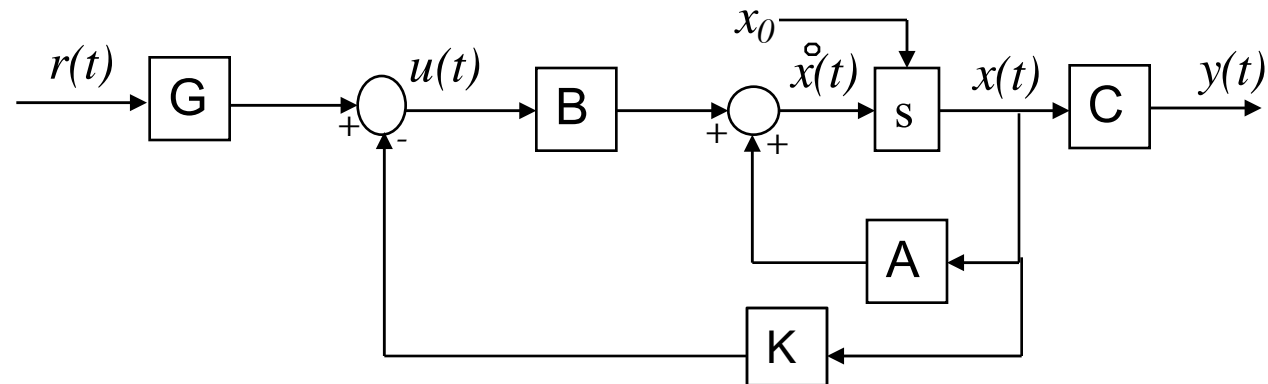
$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{G}\mathbf{r}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Closed-loop system

► Why has $\mathbf{u}(t)$ been re-defined in this way?

$r(t)$ is called the reference input. It is set to the desired value of $y(t)$.



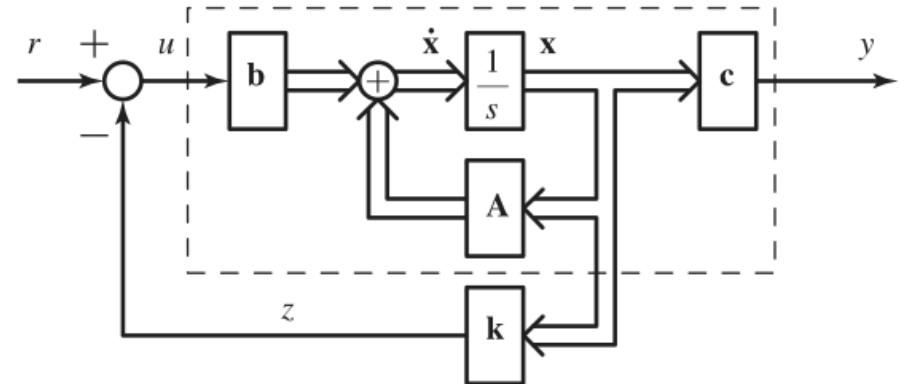
- $\mathbf{u}(t)$ now plays a similar role to an error signal in classical control. It gives the difference between the reference input $r(t)$ and the system state, scaled through \mathbf{K} .
 - However, the feedback now comes from $\mathbf{x}(t)$ not $\mathbf{y}(t)$.

Application to SISO Systems

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) + \mathbf{G}r(t)$$

► If the system is SISO:

- $u(t)$ is a scalar
- $r(t)$ is a scalar
- \mathbf{K} is a $1 \times n$ row vector
- \mathbf{G} is a scalar



► We can re-write $\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) + \mathbf{G}r(t)$ as:

$$u(t) = - \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + Gr(t) = -k_1x_1(t) - k_2x_2(t) - \cdots - k_nx_n(t) + Gr(t)$$

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{BK})\mathbf{x}(t) + \mathbf{B}\mathbf{Gr}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

► **Feedback Gain Formula for Controller Canonical Form (CCF)**

The coefficient matrices for CCF are given below:

$$\mathbf{A}_{CCF} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \quad \mathbf{B}_{CCF} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

its characteristic polynomial (transfer function denominator) is written as:

$$s^n + a_{n-1}s^{n-1} + \dots + a_2s^2 + a_1s + a_0$$

For the single-input case, the gain matrix \mathbf{K} is reduced to a feedback gain vector denoted by:

$$\mathbf{K}_{CCF} = [k_0 \ k_1 \ k_2 \ \dots \ k_{n-1}]$$

Thus, the closed-loop system dynamics matrix is:

$$\mathbf{A}_{CCF} - \mathbf{B}_{CCF}\mathbf{K}_{CCF} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 - k_0 & -a_1 - k_1 & -a_2 - k_2 & \dots & -a_{n-1} - k_{n-1} \end{bmatrix}$$

with char. poly: $s^n + (a_{n-1} + k_{n-1})s^{n-1} + \dots + (a_2 + k_2)s^2 + (a_1 + k_1)s + (a_0 + k_0)$

Closed-loop Eigenvalue Placement

The characteristic polynomial for the compensated system:

$$s^n + (a_{n-1} + k_{n-1})s^{n-1} + \cdots + (a_2 + k_2)s^2 + (a_1 + k_1)s + (a_0 + k_0)$$

The following represents the desired characteristic polynomial:

$$s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_2s^2 + \alpha_1s + \alpha_0$$

These alpha coefficients would be arrived at by looking at the constraints imposed by the problem (e.g. specifying a desired %OS and T_s).

We determine \mathbf{K}_{CCF} by pairing, term-by-term, the two polynomials above (the actual and the desired) and solving for the k values.

$$\alpha_0 = a_0 + k_0 \quad \alpha_1 = a_1 + k_1 \quad \alpha_2 = a_2 + k_2 \quad \cdots \quad \alpha_{n-1} = a_{n-1} + k_{n-1}$$

which yields:

$$\mathbf{K}_{CCF} = [(\alpha_0 - a_0) \quad (\alpha_1 - a_1) \quad \cdots \quad (\alpha_{n-1} - a_{n-1})]$$

Dynamic Response Shaping

- ▶ We are interested in shaping the transient response (often the step response), by modifying such characteristics as rise time (T_r), peak time (T_p), percent overshoot (%OS), and settling time (T_s)
- ▶ Second-order dominant systems are frequently used as approximations in the design process
 - That is if the system is 3rd order or higher, it is approximated as 2nd order
 - If the system is 1st or 2nd order then it is treated as such
- ▶ Lets have a quick review of the characteristics of 1st and 2nd order systems:
 - [Time response notes from 5821]

Summary of 1st and 2nd Order System Characteristics

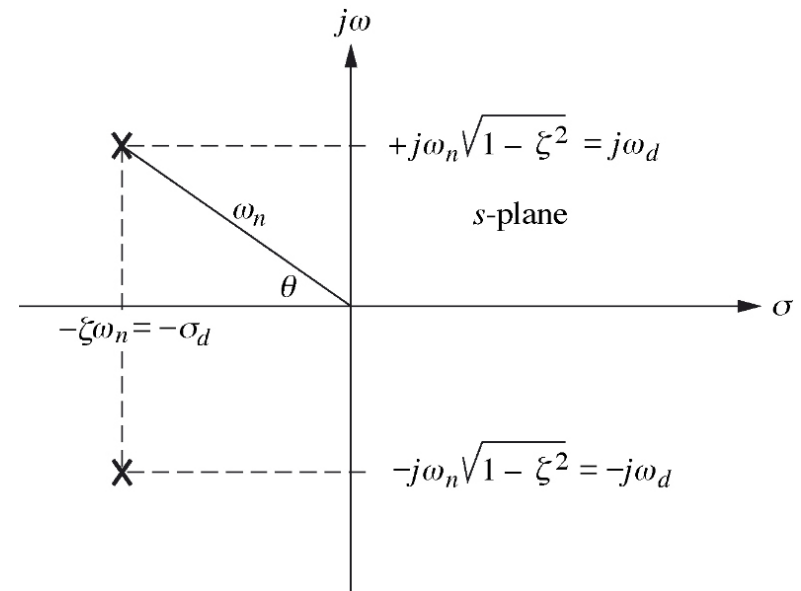
First-Order System

$$G(s) = \frac{a}{s + a} \quad T_r = \frac{2.2}{a} \quad T_s = \frac{4}{a}$$

Second-Order System

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

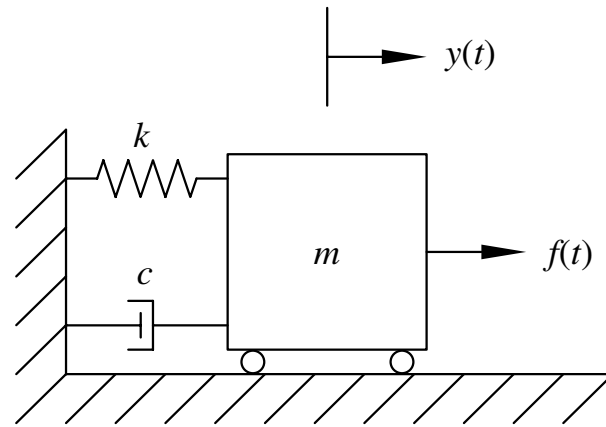
$$\cos\theta = \zeta$$



$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} \quad T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d} \quad \%OS = e^{(-\zeta\pi/\sqrt{1-\zeta^2})} \times 100 \quad \zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

Example 1

- Consider again the following mechanical system:



State variables:

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k}{m} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$H(s) = \frac{\frac{k}{m}}{s^2 + \frac{c}{m}s + \frac{k}{m}}$$

$$H(s) = \frac{\frac{k}{m}}{s^2 + \frac{c}{m}s + \frac{k}{m}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- We set specific values of the constants and compare with the standard 2nd order transfer function to obtain:

$$m = 1 \text{ kg}, c = 1 \text{ N-s/m}, \text{ and } k = 10 \text{ N/m.}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10}{1}} = 3.16 \text{ rad/s}$$

$$\xi = \frac{c}{2\sqrt{km}} = \frac{1}{2\sqrt{10(1)}} = 0.158$$

Eigenvalues:

$$\lambda_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$$\lambda_{1,2} = -0.5 \pm 3.12i$$

Step Response:

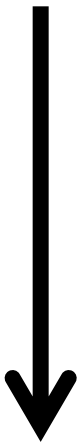
$$y(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1 - \xi^2}} \sin(\omega_d t + \theta)$$

$$y(t) = 1 - 1.01e^{-0.5t} \sin(3.12t + 80.9^\circ)$$

Calculated characteristics:

$T_s = 8$ seconds

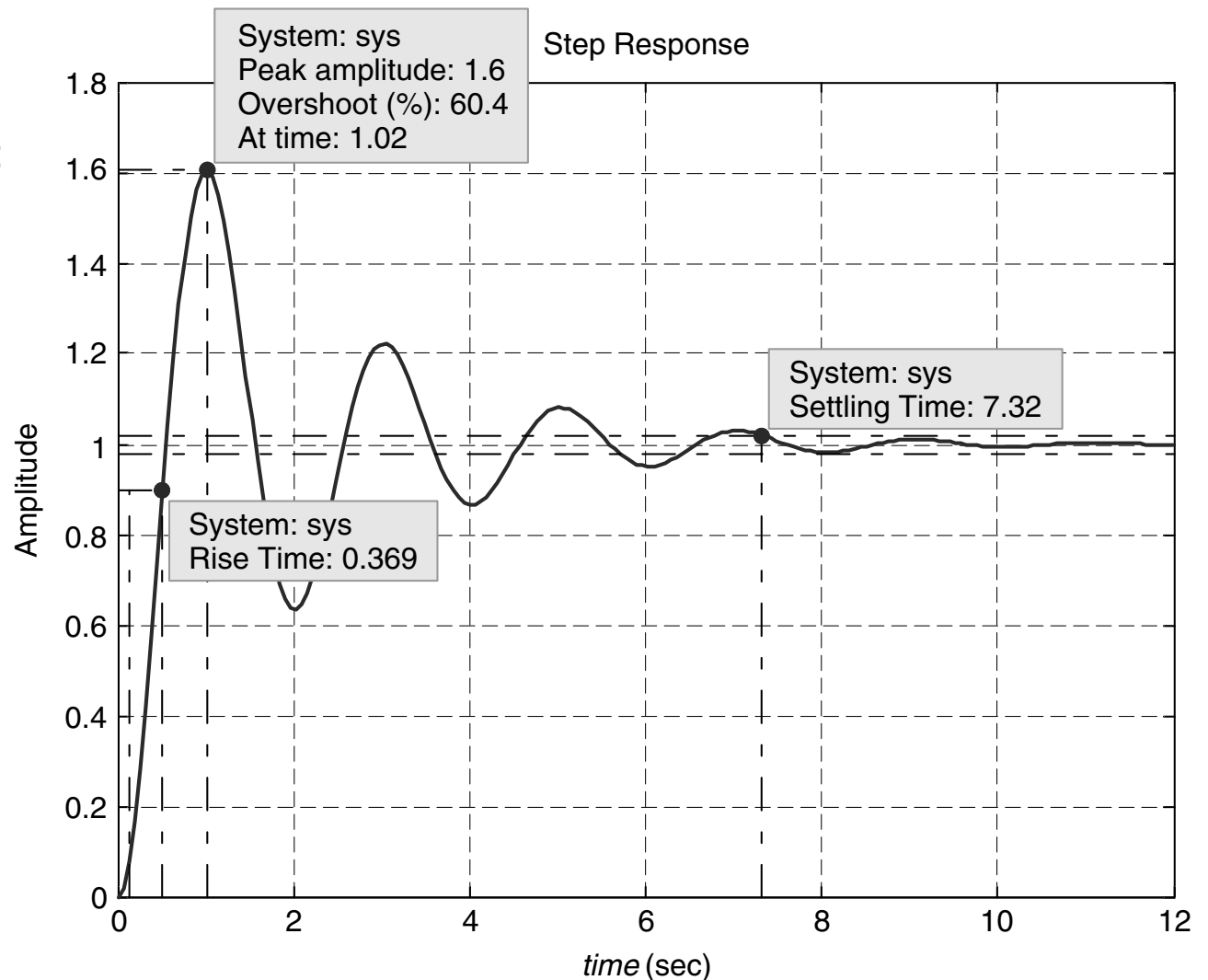
%OS = 60.46%



Both of these characteristics can be reduced! Desired:

$T_s = 2$ seconds

%OS = 4 %



In Matlab:

- Generate LTI system **sys** using **tf** or **ss**
- Generate step response with **step(sys)**
- Right-click on plot and select “Characteristics” then the characteristic of interest
- Click on generated point to see the value selected.

- We have the following desired characteristics:
 - $T_s = 2$ seconds
 - % OS = 4%
- From these we can determine the desired 2nd order system parameters

$$\xi' = \frac{\left| \ln \left(\frac{PO}{100} \right) \right|}{\sqrt{\pi^2 + \left[\ln \left(\frac{PO}{100} \right) \right]^2}} = \frac{\left| \ln \left(\frac{4}{100} \right) \right|}{\sqrt{\pi^2 + \left[\ln \left(\frac{4}{100} \right) \right]^2}} = 0.716$$

$$\omega'_n = \frac{4}{\xi' t_s} = 2.79 \text{ rad/s}$$

- Desired characteristic polynomial:

$$\lambda^2 + 2\xi' \omega'_n \lambda + \omega_n'^2 = \lambda^2 + 4\lambda + 7.81$$

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned}$$

Open-loop system

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) + \mathbf{G}\mathbf{r}(t)$$

$$\begin{aligned}\dot{\mathbf{x}}(t) &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{G}\mathbf{r}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned}$$

Closed-loop system

- Desired characteristic polynomial:

$$\lambda^2 + 2\xi'\omega_n'\lambda + \omega_n'^2 = \lambda^2 + 4\lambda + 7.81$$

- Open-loop (i.e. original) characteristic polynomial:

$$\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 = \lambda^2 + \lambda + 10$$

- $\mathbf{A} - \mathbf{B}\mathbf{K}$ matrix for the compensated system:

$$\mathbf{A} - \mathbf{B}\mathbf{K} = \begin{bmatrix} 0 & 1 \\ -10 - k_0 & -1 - k_1 \end{bmatrix}$$

- Characteristic polynomial for $\mathbf{A} - \mathbf{B}\mathbf{K}$:

$$\lambda^2 + (1 + k_1)\lambda + (10 + k_0)$$

- So we need $k_1 = 3$ and $k_0 = -2.19$

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned}$$

Open-loop system

$$A = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, C = [1 \quad 0]$$

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) + \mathbf{G}r(t)$$

$$\begin{aligned}\dot{\mathbf{x}}(t) &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{G}r(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned}$$

Closed-loop system

$$A - BK = \begin{bmatrix} 0 & 1 \\ -10 - k_0 & -1 - k_1 \end{bmatrix}$$

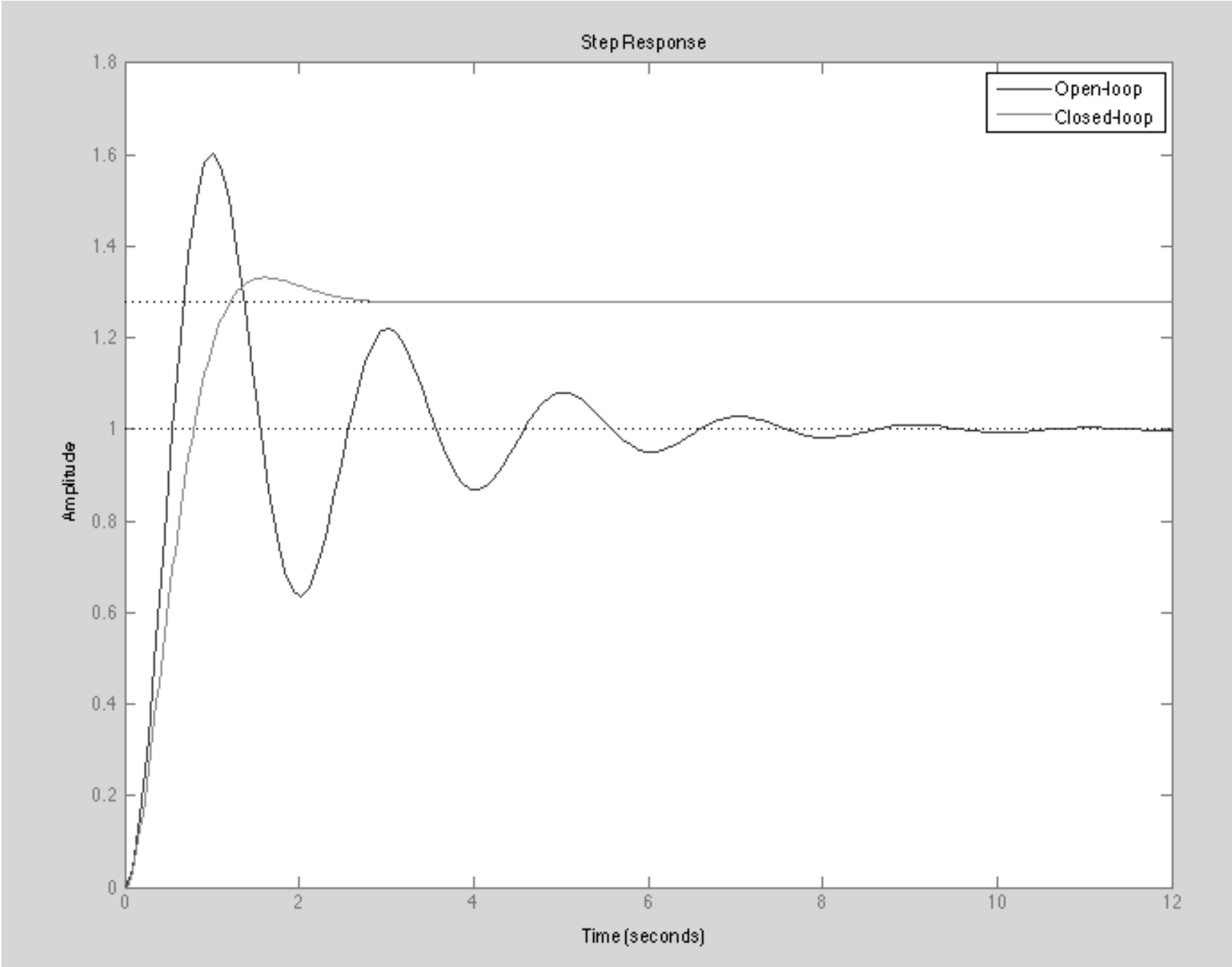
Are we done? Not yet. Assuming $G = 1$, what transfer functions would we obtain?

Obtain transfer functions from A , B , and C (open-loop) or $A-BK$, BG , and C (closed-loop):

$$H_{open}(s) = C(sI - A)^{-1}B + D = \frac{10}{s^2 + s + 10}$$

$$H_{closed}(s) = C(sI - A + BK)^{-1}BG + D = \frac{10}{s^2 + 4s + 7.81}$$

Not in pure 2nd order form so the step response will not go to 1



$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned}$$

Open-loop system

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) + \mathbf{G}\mathbf{r}(t)$$

$$\begin{aligned}\dot{\mathbf{x}}(t) &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{G}\mathbf{r}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned}$$

Closed-loop system

$$H_{closed}(s) = C(sI - A + BK)^{-1}BG + D = \frac{10}{s^2 + 4s + 7.81}$$

- Consider the step response for this system:

$$Y(s) = H(s) \frac{1}{s}$$

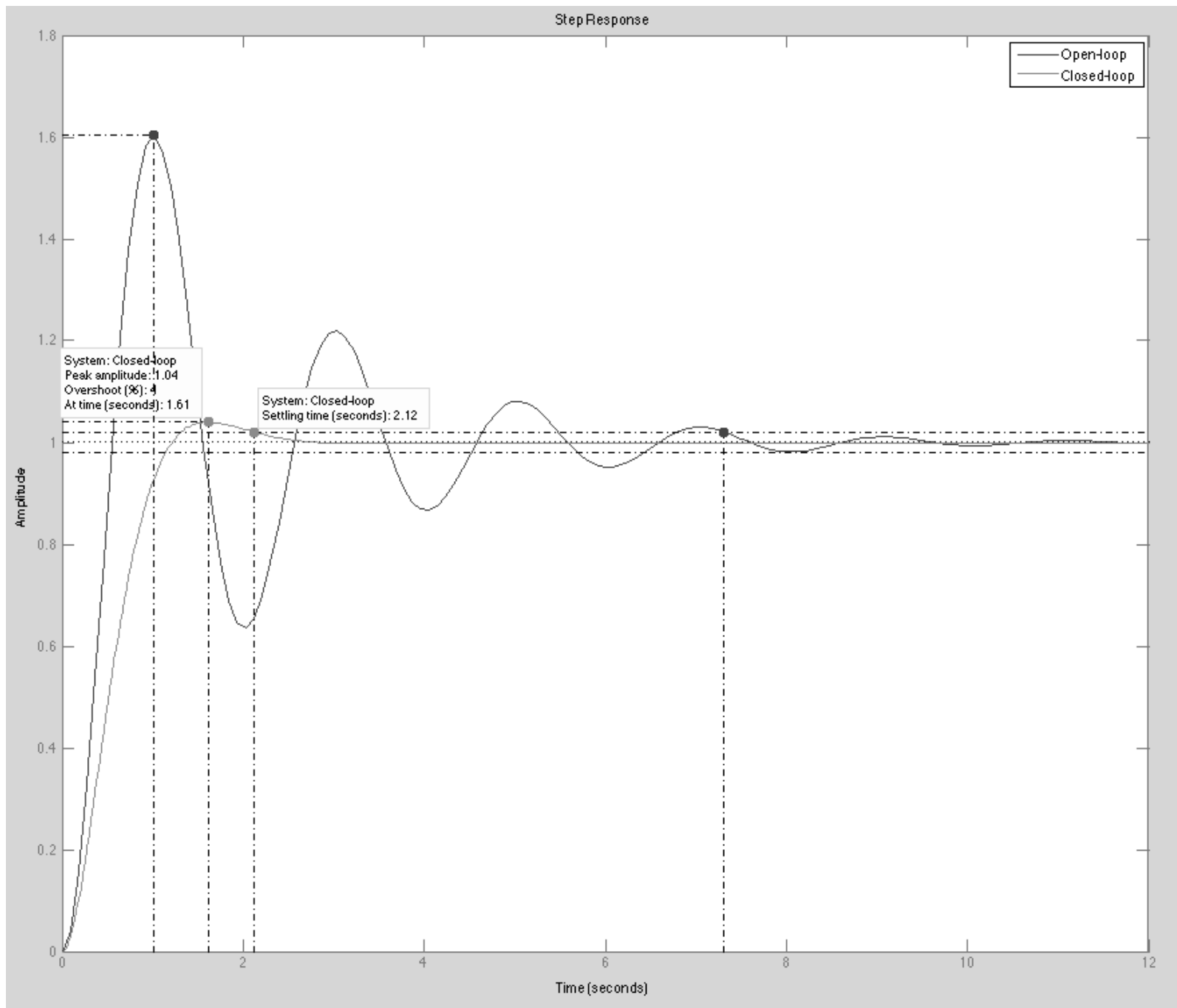
- What's the steady-state value? We can use the **final-value theorem** to find out:

$$\text{The Final Value Theorem: } f(\infty) = \lim_{s \rightarrow 0} sF(s)$$

$$y(\infty) = \lim_{s \rightarrow 0} H(s) = \frac{10}{7.81} \quad H(s=0) \text{ also known as } \mathbf{DC \text{ gain}}$$

- If we want a steady-state value of $y(1) = 1$ then set $G = 7.81 / 10$

$$H(s) = \frac{7.81}{s^2 + 4s + 7.81}$$



Success! %OS = 4 as desired. However, $T_s = 2.12$ which is a little larger than 2. Recall that our formula for T_s is actually based on an approximation, so this is a good result