Stability

ENGI 7825: Control Systems II

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Introduction

- Recall that the state response of an LTI system consists of two parts:

$$\mathbf{x}(t) = \underbrace{e^{\mathbf{A}(t-t_0)}\mathbf{x_0}}_{\text{zero-input response: } x_{zi}(t)} + \underbrace{\int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau}_{x_{zs}(t)\text{: zero-state response}}$$

$$\mathbf{y}(t) = \underbrace{\mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{x_0}}_{\text{zero-input output: } y_{zi}(t)} + \underbrace{\int_{t_0}^{t} \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau + \mathbf{D}\mathbf{u}(t)}_{y_{zs}(t)\text{: zero-state output}}$$

- Stability analysis has two corresponding aspects:
  - Internal stability: Whether $x_{zi}(t)$ stays bounded
  - Bounded-input bounded stability: Whether $y_{zs}(t)$ stays bounded for a bounded input

Whether the zero-input state response stays bounded

INTERNAL STABILITY

Internal Stability

- Here, we assume u(t) = 0 and focus on the system's behaviour on its own
- The fundamental equation is

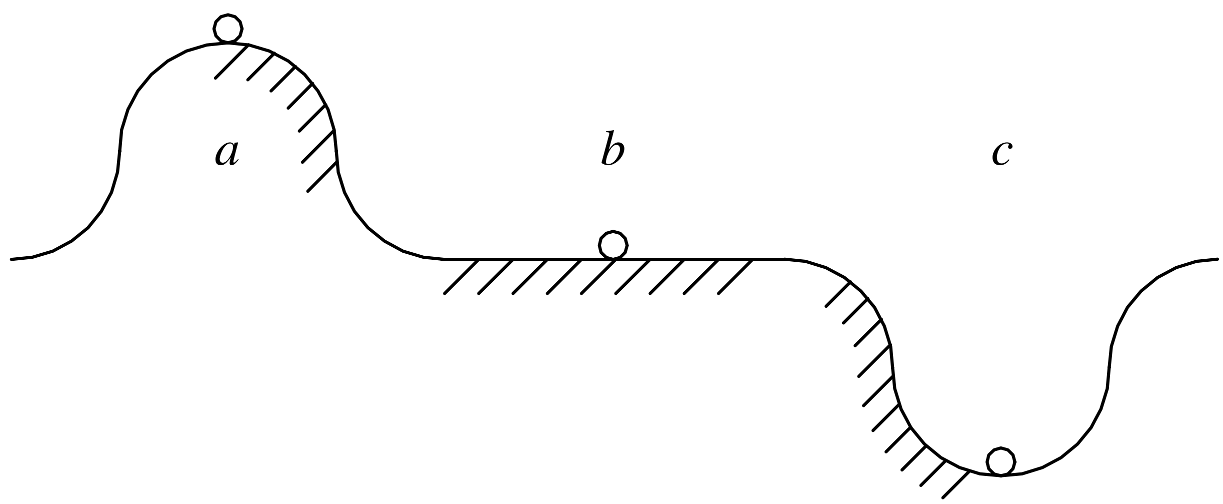
$$\dot{x}(t) = Ax(t) \qquad x(0) = x_0$$

- Or for a nonlinear system

$$\dot{x}(t) = f[x(t)] \qquad x(0) = x_0$$

- An **equilibrium state** $\tilde{x}$ is a particular state vector at which the derivative equals 0

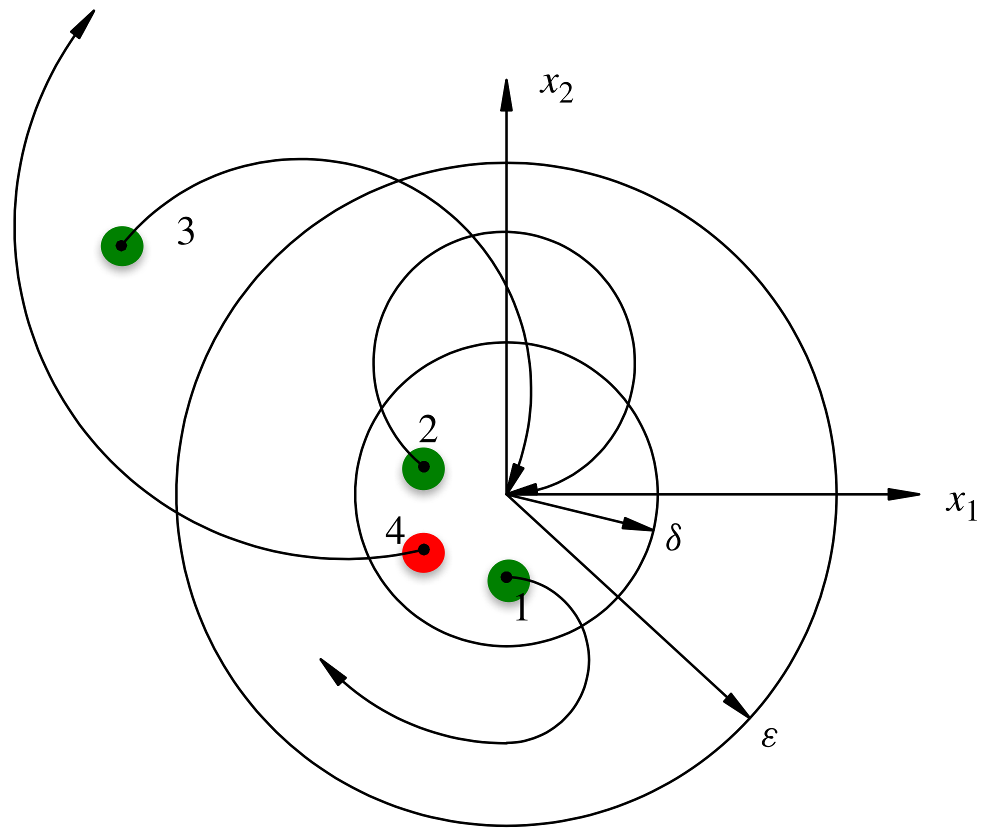
- Depicted below are equilibrium states a, b and c:
  - **a is unstable:** even a tiny movement will move the state away from equilibrium
  - **b is stable:** a small movement will move the state a small distance
  - **c is asymptotically stable:** a small movement will move the state, but it will eventually return to the original point.



Stable and Unstable

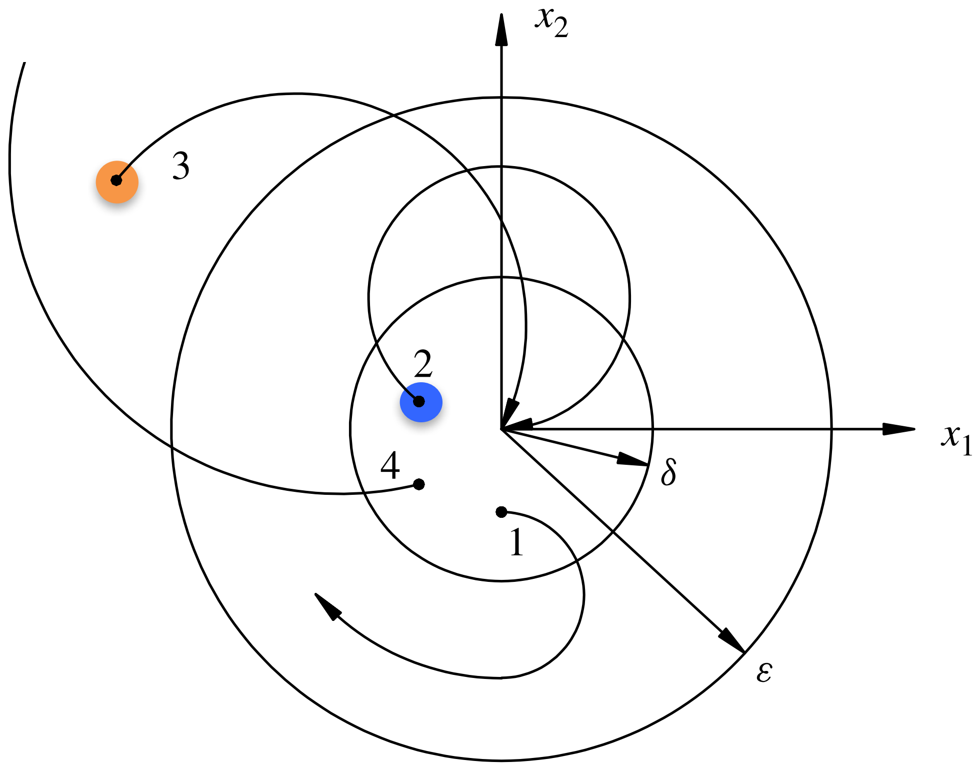
Definition 6.1 *The equilibrium state* $\tilde{x} = 0$ *of Equation* (6.2) *is*

- ***Stable*** *if, given any* $\varepsilon > 0$ *there corresponds a* $\delta > 0$ *such that* $\|x_0\| < \delta$ *implies that* $\|x(t)\| < \varepsilon$ *for all* $t \geq 0$.
- ***Unstable*** *if it is not stable.*



Asymptotically Stable

- ***Asymptotically stable*** *if it is stable and it is possible to choose* $\delta > 0$ *such that* $\|x_0\| < \delta$ *implies that* $\lim_{t\to\infty} \|x(t)\| = 0$*. Specifically, given any* $\varepsilon > 0$*, there exists* $T > 0$ *for which the corresponding trajectory satisfies* $\|x(t)\| \le \varepsilon$ *for all* $t \ge T$*.*
- ***Globally asymptotically stable*** *if it is stable and* $\lim_{t\to\infty} \|x(t)\| = 0$ *for any initial state. Specifically, given any* $M > 0$ *and* $\varepsilon > 0$*, there exists* $T > 0$ *such that* $\|x_0\| < M$ *implies that the corresponding trajectory satisfies* $\|x(t)\| \le \varepsilon$ *for all* $t \ge T$*.*



Assumptions

- If u(t) = 0 and t_0 = 0, the state response is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x_0}$$

- We will assume the following (each statement implies the others):
  – A has n linearly independent eigenvectors
  – A has n distinct eigenvalues
  – A can be diagonalized

$$A = VDV^{-1}$$

- We expand the definition of the matrix exponential to incorporate VDV^{-1} in place of A:

Using

$$A = VDV^{-1} \qquad A^k = VD^kV^{-1}$$

$$\begin{aligned} e^{At} &= I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \\ &= VV^{-1} + VDV^{-1} + \frac{VD^2V^{-1}t^2}{2!} + \frac{VD^3V^{-1}t^3}{3!} + \cdots \\ &= V\left(I + Dt + \frac{D^2t^2}{2!} + \frac{D^3t^3}{3!} + \cdots\right)V^{-1} \\ &= Ve^{Dt}V^{-1} \end{aligned}$$

- What does e^{Dt} look like?

$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & & 0 \\ & & \ddots & \\ 0 & \cdots & 0 & e^{\lambda_n t} \end{bmatrix}$$

- We can use this to re-write x(t)

$$\begin{aligned} x(t) &= e^{At} x_0 \\ &= V e^{Dt} V^{-1} x_0 \end{aligned}$$

- V, V^{-1}, and x_0 are constant matrices and vectors. Therefore each row of x(t) is just some linear combination of terms involving the diagonals of e^{Dt}

$$x_i(t) = a e^{\lambda_1 t} + b e^{\lambda_2 t} + \cdots$$

$$x_i(t) = ae^{\lambda_1 t} + be^{\lambda_2 t} + \cdots$$

- The exact state response requires all these constants (a, b, ...) to be known, but for stability analysis we just want to know if the expression will grow without bound
- Clearly, if the eigenvalues of A (λ_1, λ_2, ...) are real numbers then we just have a sum of pure exponentials
  - If any $\lambda_i > 0$ then the system is unstable
  - If all $\lambda_i < 0$ then the system is asymptotically stable
  - If n – 1 eigenvalues are negative, and just one eigenvalue is zero then the system is stable, but not asymptotically stable

Complex Eigenvalues

- The eigenvalues of A will often be complex, even if A is purely real
- Consider $e^{\lambda t}$ and let $\lambda = a + i\,b$

$$
\begin{aligned}
e^{\lambda t} = e^{(a+ib)t} &= e^{at} e^{ibt} \\
&= e^{at} \left(\cos bt + i \sin bt\right) \\
|e^{\lambda t}| &= e^{at}
\end{aligned}
$$

- This magnitude decays for $a < 0$, stays constant for $a = 0$, and grows (explodes!) for $a > 0$
- Hence, for stability all we care about is $\mathrm{Re}\{\lambda\}$

The Eigenvalue Test for Internal Stability

- If our assumption of n distinct eigenvalues holds, then we have the following eigenvalue test for internal stability:
  - The system is stable if all eigenvalues have a non-positive real part
    - If one eigenvalue is zero then there is a constant non-decaying term
    - Otherwise, if all eigenvalues are strictly negative we have asymptotic stability
  - If any eigenvalue has a positive real part then the system is unstable

- The same holds for repeated eigenvalues, except for repeated eigenvalues with zero real part. In this case a more sophisticated test is required (out of our scope).

Connection between Poles and Eigenvalues

- The poles of a system's transfer function will be eigenvalues of A
  - Caveat: Its possible that some of the eigenvalues of A will not be poles of the transfer function due to pole-zero cancellation.
    - (We'll see an example of this later)

- Let H(s) be a general 2nd order transfer function

$$H(s) = \frac{b}{s^2 + as + b} \qquad \text{poles:} \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

- The corresponding A matrix

$$A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}$$

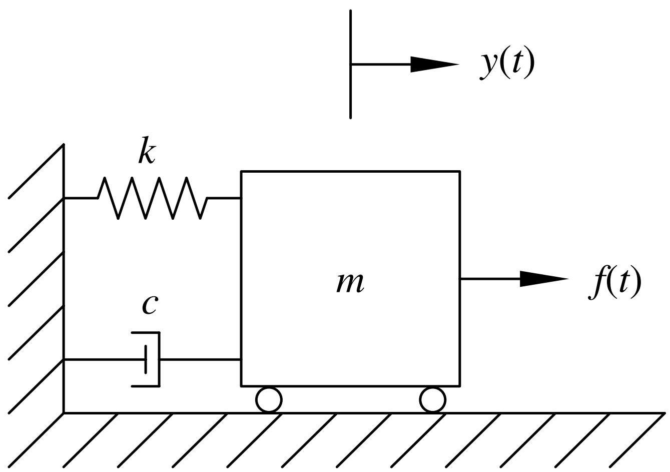
- Lets work out its eigenvalues

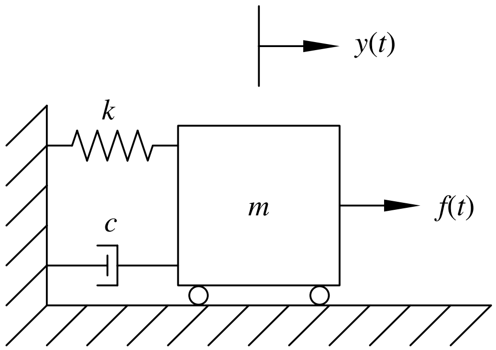
$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det\left(\begin{bmatrix} -\lambda & 1 \\ -b & -a-\lambda \end{bmatrix}\right) &= 0 \\ \lambda^2 + a\lambda + b &= 0 \\ \lambda &= \frac{-a \pm \sqrt{a^2 - 4b}}{2} \end{aligned}$$

The Characteristic Polynomial

Energy-Based Analysis

- Recall that state variables are always associated with energy storage elements
- A stable system will dissipate or maintain energy
  - If the system's energy always dissipates down to 0 then it is asymptotically stable
- If the energy in the system actually increases (without any applied input) then the system is unstable
- Consider the following simple mechanical system...



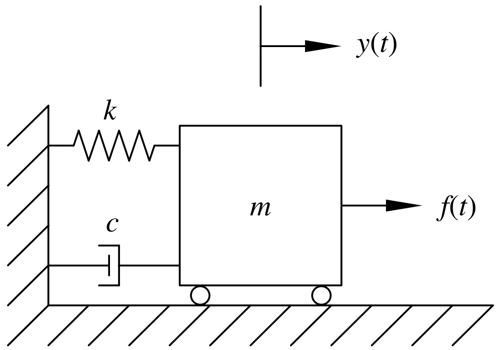


$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}$$

- The state variable $x_1(t)$ represents the displacement of the mass. The system stores energy in two ways:
  - Potential energy in the spring: ½ k x_1^2
  - Kinetic energy in the moving mass: ½ m x_2^2
  - Total energy:

$$E(x_1, x_2) = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$$

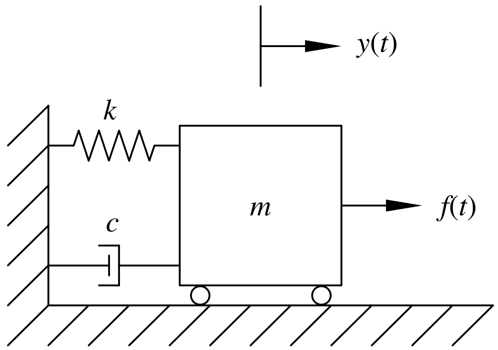
- Work out the derivative of E with respect to time...



$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \qquad E(x_1, x_2) = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$$

$$\begin{aligned} \frac{d}{dt}E[x_1(t), x_2(t)] &= \frac{d}{dt}\left[\frac{1}{2}kx_1^2(t) + \frac{1}{2}mx_2^2(t)\right] \\ &= kx_1(t)\dot{x}_1(t) + mx_2(t)\dot{x}_2(t) \\ &= kx_1(t)\Big[x_2(t)\Big] + mx_2(t)\left[-\frac{k}{m}x_1(t) - \frac{c}{m}x_2(t)\right] \\ &= -cx_2^2(t) \end{aligned}$$

- Lets try adjusting the damping coefficient, c



$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}$$

$$E(x_1, x_2) = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$$

$$\dot{E}[x_1(t), x_2(t)] = -cx_2^2(t)$$

- Zero damping: c = 0
  - dE/dt = 0 which means that the total energy is constant
  - Energy goes back and forth between the moving mass and the spring, but is never lost (or gained)
  - $\lambda_1, \lambda_2 = \pm j\,3.16$ (for m = 1kg, k = 10 N/m)
  - The system oscillates sinusoidally
  - The system is stable, but not asymptotically stable

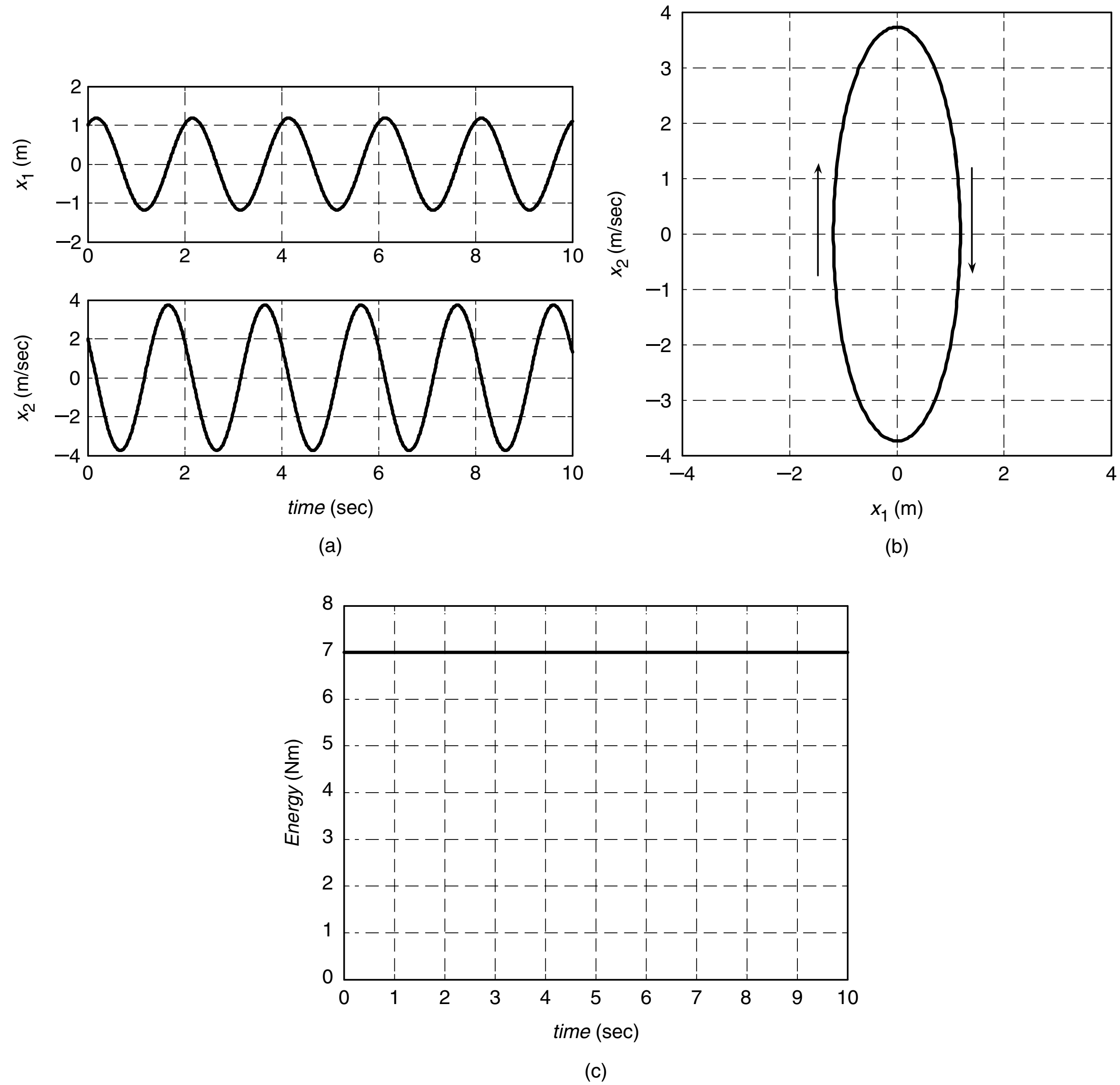
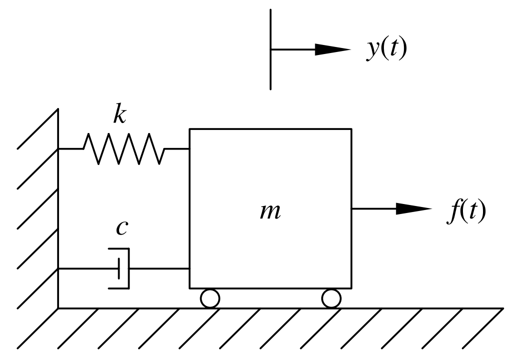


FIGURE 6.4 (*a*) State-variable responses; (*b*) phase portrait; (*c*) energy response for a marginally-stable equilibrium.

"Phase portrait": Shows system trajectory in state space

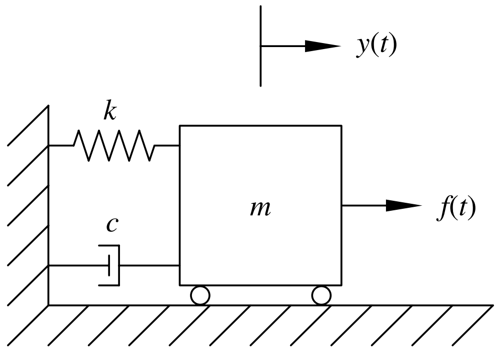


Specific values:

$m = 1$ kg

$k = 10$ N/m

$x_0 = [1, 2]^T$



$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}$$

$$E(x_1, x_2) = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$$

$$\dot{E}[x_1(t), x_2(t)] = -cx_2^2(t)$$

- Positive damping: c = 1
  - dE/dt < 0 which means that energy strictly decreases
  - λ_1, λ_2 = -0.5 ± j 3.12
  - Asymptotically stable:
    - State response is an exponentially decaying sinusoid

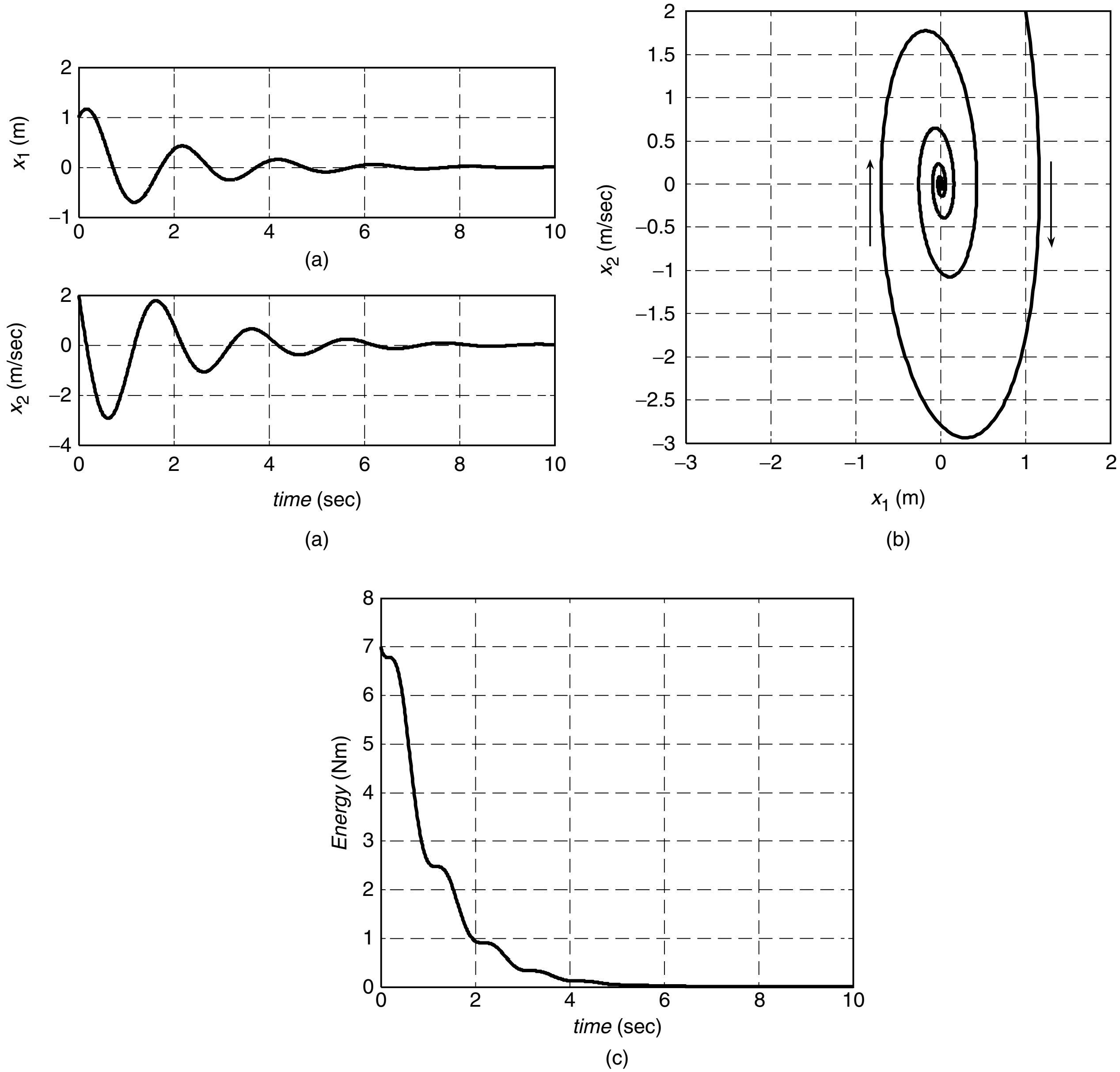
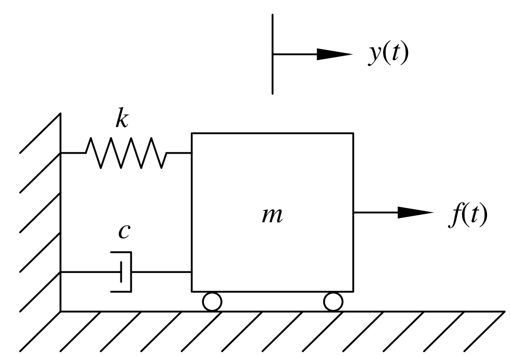
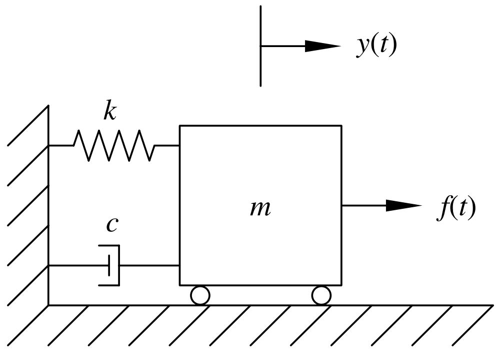


FIGURE 6.5 (*a*) State-variable responses; (*b*) phase portrait; (*c*) energy response for an asymptotically-stable equilibrium.

Convergence to $[0, 0]^T$, hence asymptotically stable



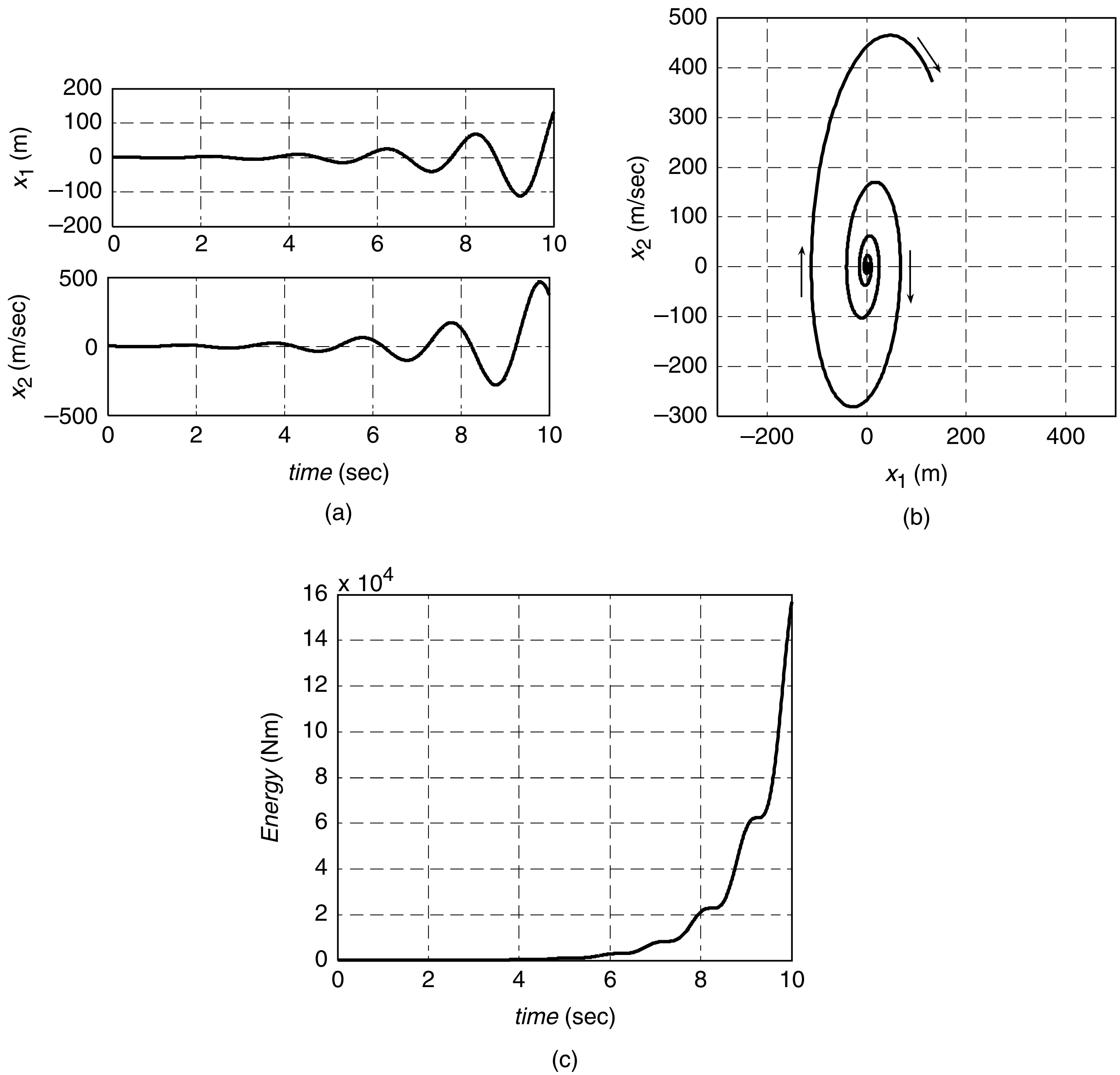


$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}$$

$$E(x_1, x_2) = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$$

$$\dot{E}[x_1(t), x_2(t)] = -cx_2^2(t)$$

- Negative damping: c = -1
  - Not clear what this means physically: A powered damper?
  - dE/dt > 0 which means that energy strictly **increases**
  - λ_1, λ_2 = 0.5 ± j 3.12
  - Unstable:
    - State response is an exponentially **growing** sinusoid



This unstable system is on its way to some sort of break down!

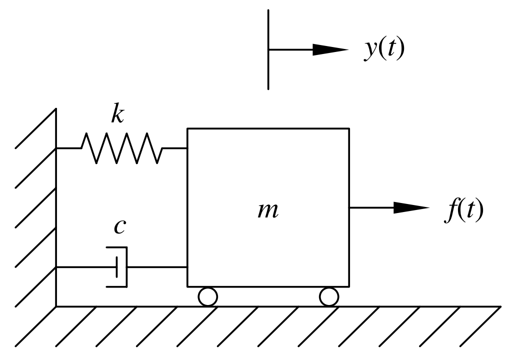


FIGURE 6.6 (*a*) State-variable responses; (*b*) phase portrait; (*c*) energy response for an unstable equilibrium.

Whether the zero-state response is bounded for a bounded input

BOUNDED-INPUT BOUNDED-OUTPUT STABILITY

Bounded-Input Bounded-Output (BIBO) Stability

- Recall that the output of an LTI system consists of two parts:

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{x_0} + \int_{t_0}^{t} \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau + \mathbf{D}\mathbf{u}(t)$$

zero-input output: $y_{zi}(t)$

$y_{zs}(t)$: zero-state output

- BIBO stability concerns $y_{zs}(t)$
- As you have seen BIBO stability in other courses we will be brief:
  - A system is BIBO stable if its impulse response has a finite sum

Test for BIBO Stability

Theorem 6.6 *The linear state equation* (6.1) *is bounded-input, bounded-output stable if and only if the impulse response matrix* $H(t) = Ce^{At}B + D\delta(t)$ *satisfies*

$$\int_0^\infty \|H(\tau)\| d\tau < \infty$$

- This means that H(t) is absolutely integrable
- There is a strong relationship between internal stability and BIBO stability (both involve e^{At})
  - If a system is asymptotically stable, it is BIBO stable
  - However, the opposite is not necessarily true...

Example 6.3 Consider the following two-dimensional state equation:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [\,0 \quad 1\,] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The characteristic polynomial is

$$\begin{aligned} |sI - A| &= \begin{vmatrix} s & -1 \\ -1 & s \end{vmatrix} \\ &= s^2 - 1 \\ &= (s+1)(s-1) \end{aligned} \qquad \lambda_{1,2} = -1, +1$$

Not asymptotically stable... In fact, unstable!

$$e^{At} = \begin{bmatrix} \frac{1}{2}(e^t + e^{-t}) & \frac{1}{2}(e^t - e^{-t}) \\ \frac{1}{2}(e^t - e^{-t}) & \frac{1}{2}(e^t + e^{-t}) \end{bmatrix}$$

$$
\begin{aligned}
H(s) &= C(sI - A)^{-1}B \\
&= [0 \quad 1]\begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}^{-1}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
&= [0 \quad 1]\frac{\begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix}}{s^2 - 1}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
&= \frac{s-1}{(s+1)(s-1)} \\
&= \frac{1}{(s+1)}
\end{aligned}
$$

Pole-Zero Cancellation

$$h(t) = e^{-t},\ t \geq 0$$

Satisfies the test for BIBO stability: $\int_0^\infty |h(\tau)|d\tau = 1$

So the system is unstable, but BIBO stable? Why?

Lesson: Don't cancel poles and zeros when testing for stability