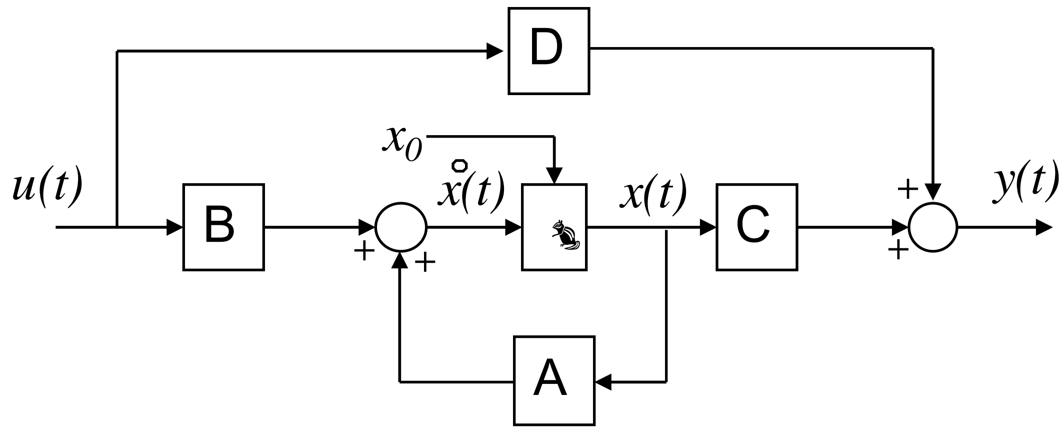

ENGI 7825: Control Systems II

The State-Space Representation: Part 4: Linearization

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Linearity

- A function, f, is linear if it exhibits the following properties:
  - Additivity: $f(p + q) = f(p) + f(q)$
    - Also known as the superposition principle
  - Homogeneity: $f(\alpha p) = \alpha\, f(p)$
- Consider the function $f(x) = mx + b$:
  - Additivity?
    - $f(p) = mp + b$
    - $f(q) = mq + b$
    - $f(p + q) = m(p + q) + b = mp + mq + b$
    - This is not equal to $f(p) + f(q)$!
    - So the equation for a straight line is not linear!! Although it would be linear if $b = 0$.

Linear Differential Equations

- For a differential equation to be linear, it must be possible to form linear combinations of solutions which are also solutions. This is the case for any DE of the following form:

$$a_n x^{(n)}(t) + a_{n-1} x^{(n-1)}(t) + \cdots + a_0 x(t) = u(t)$$

- The key point is that we have nothing but derivatives multiplied by constant coefficients on the left.
- The SS representation can represent multiple linear DE's together.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

- Our ability to describe the system in SS form implies a linear system.

Linearization of nonlinear systems

- The SS representation for a nonlinear time-varying system is as follows:

$$\dot{x}(t) = f[x(t), u(t), t] \qquad x(t_0) = x_0$$
$$y(t) = h[x(t), u(t), t]$$

- A system is nonlinear if it cannot be written in the standard SS form.
- There exist techniques to solve some nonlinear state equations, but they will not be studied in this course.
- We will use a first-order Taylor series expansion to linearize such a system about a particular operating point.

Taylor series expansion (review)

The Taylor series expansion gives the value of a function, *f*, at *t* from its value at *a* and its derivatives evaluated at *a:*

$$f(t) = f(a) + \frac{f'(a)}{1!}(t-a) + \frac{f''(a)}{2!}(t-a)^2 + \frac{f^{(3)}(a)}{3!}(t-a)^3 + \cdots$$

The higher-order terms are typically small, so we often make a first-order approximation by eliminating them:

$$f(t) \approx f(a) + \frac{f'(a)}{1!}(t-a)$$

The point a is called the operating point and f(a) is the **nominal value**.

1D Example

- Find a linear approximation for the function $f(t) = t^2$ at the operating point $t = 1$
  - COVERED ON BOARD
- The resulting function, $2t – 1$, is not linear!
- However, we can define a new deviation variable $f_\delta(t)$ with respect to the nominal value of the function at $t = 1$

$$
\begin{aligned}
f_\delta(t) &= f(t) - f_n(t) \\
&= (2t - 1) - (1) \\
&= 2t - 2
\end{aligned}
$$

- This is still not linear, but we can make it linear by defining a new time variable, t_δ defined as $t_\delta = t – 1$. When we substitute for $t = t_\delta + 1$ we get,

$$f_\delta(t_\delta) = 2(t_\delta + 1) - 2 = 2t_\delta$$

2D Example

- Linear approximation for a two variable function f(x, y)
  - A linear approximation for f(x,y) about (a,b), is obtained with a first-order multivariate Taylor series expansion

$$f(x,y) \approx f(a,b) + \left.\frac{\partial f}{\partial x}\right|_{(a,b)} (x-a) + \left.\frac{\partial f}{\partial y}\right|_{(a,b)} (y-b)$$

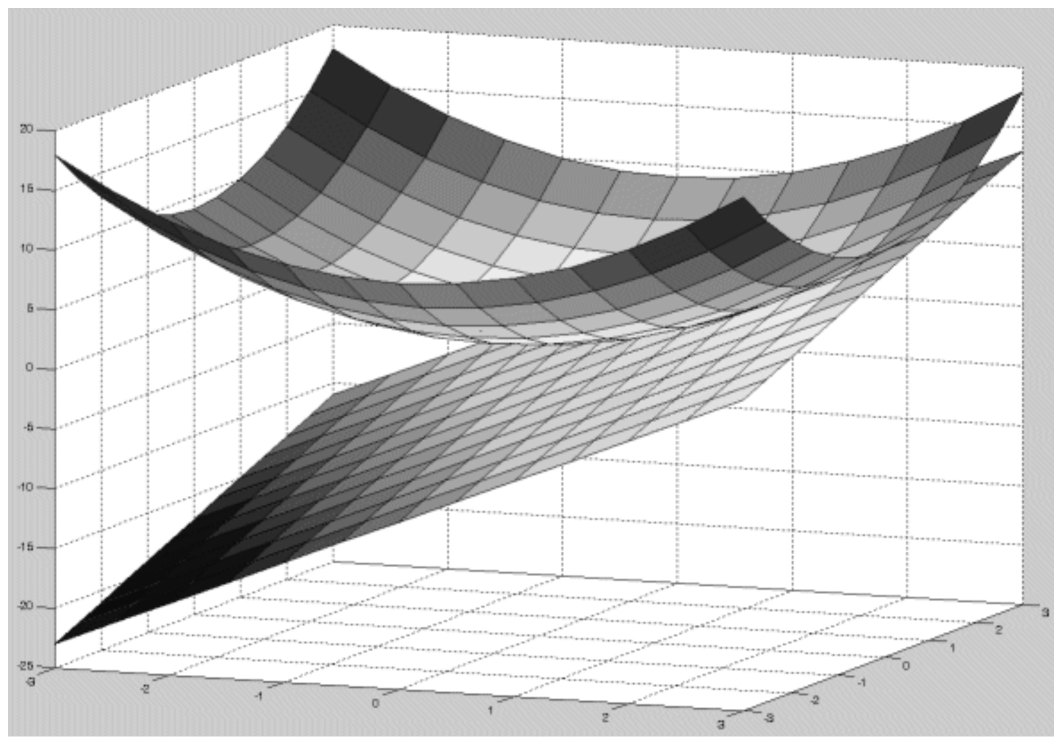
Example: Find the linearization of $f(x,y)=x^2+y^2$ about point (1,2)

$$f(1,2)=5$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,2)} = \left.2x\right|_{(1,2)} = 2$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,2)} = \left.2y\right|_{(1,2)} = 4$$

$$f(x,y) = 2x + 4y - 5$$



Linearization of nonlinear systems

- Nonlinear, time-varying systems can be represented by state equations:

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left[\mathbf{x}(t), \mathbf{u}(t), t\right]$$
$$\mathbf{y}(t) = \mathbf{h}\left[\mathbf{x}(t), \mathbf{u}(t), t\right] \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

where **f** and **h** are continuously differentiable functions. Linearization is obtained as follows:

  - Assume that the system operates along some nominal trajectory $x_n(t)$ while it is driven by the system input $u_n(t)$. These are called the <u>nominal state trajectory</u> and the <u>nominal input trajectory</u>, respectively.

$$\dot{\mathbf{x}}_n(t) = \mathbf{f}\left[\mathbf{x}_n(t), \mathbf{u}_n(t), t\right]$$
$$\mathbf{y}_n(t) = \mathbf{h}\left[\mathbf{x}_n(t), \mathbf{u}_n(t), t\right]$$

  - Expanding the nonlinear functions in a multivariate first-order Taylor series expansion about $[x_n(t), u_n(t), t]$ we obtain:

$$\dot{\mathbf{x}}(t) \approx \mathbf{f}\left[\mathbf{x}_n(t), \mathbf{u}_n(t), t\right] + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\left[\mathbf{x}_n(t), \mathbf{u}_n(t), t\right]\left[\mathbf{x}(t) - \mathbf{x}_n(t)\right] + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\left[\mathbf{x}_n(t), \mathbf{u}_n(t), t\right]\left[\mathbf{u}(t) - \mathbf{u}_n(t)\right]$$
$$\mathbf{y}(t) \approx \mathbf{h}\left[\mathbf{x}_n(t), \mathbf{u}_n(t), t\right] + \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\left[\mathbf{x}_n(t), \mathbf{u}_n(t), t\right]\left[\mathbf{x}(t) - \mathbf{x}_n(t)\right] + \frac{\partial \mathbf{h}}{\partial \mathbf{u}}\left[\mathbf{x}_n(t), \mathbf{u}_n(t), t\right]\left[\mathbf{u}(t) - \mathbf{u}_n(t)\right]$$

$$\dot{\mathbf{x}}(t) \approx \mathbf{f}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right] + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right]\left[\mathbf{x}(t)-\mathbf{x}_n(t)\right] + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right]\left[\mathbf{u}(t)-\mathbf{u}_n(t)\right]$$

$$\mathbf{y}(t) \approx \mathbf{h}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right] + \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right]\left[\mathbf{x}(t)-\mathbf{x}_n(t)\right] + \frac{\partial \mathbf{h}}{\partial \mathbf{u}}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right]\left[\mathbf{u}(t)-\mathbf{u}_n(t)\right]$$

- This is exactly the same as our 2-variable example, except the variables that we differentiate by are now vectors, and the corresponding derivatives are matrices!
- Lets say we have a function **f**(**v**) where **v** is a vector. The derivative of **f** is called the Jacobian matrix and is defined as follows if the output of **f** is 2-dimensional and **v** is 3-dimensional.

$$\frac{\partial \mathbf{f}}{\partial \mathbf{v}} = \begin{bmatrix} \frac{\partial f_1}{\partial v_1} & \frac{\partial f_1}{\partial v_2} & \frac{\partial f_1}{\partial v_3} \\ \frac{\partial f_2}{\partial v_1} & \frac{\partial f_2}{\partial v_2} & \frac{\partial f_2}{\partial v_3} \end{bmatrix}$$

- In general, lets say that **f** is m-dimensional and **v** is n-dimensional.

$$\frac{\partial \mathbf{f}}{\partial \mathbf{v}} = \begin{bmatrix} \frac{\partial f_1}{\partial v_1} & \cdots & \frac{\partial f_1}{\partial v_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial v_1} & \cdots & \frac{\partial f_m}{\partial v_n} \end{bmatrix}$$

$$\dot{\mathbf{x}}(t) \approx \mathbf{f}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right] + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right]\left[\mathbf{x}(t)-\mathbf{x}_n(t)\right] + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right]\left[\mathbf{u}(t)-\mathbf{u}_n(t)\right]$$
$$\mathbf{y}(t) \approx \mathbf{h}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right] + \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right]\left[\mathbf{x}(t)-\mathbf{x}_n(t)\right] + \frac{\partial \mathbf{h}}{\partial \mathbf{u}}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right]\left[\mathbf{u}(t)-\mathbf{u}_n(t)\right]$$

- We can define the following deviation variables:

$$\mathbf{x}_\delta(t) = \mathbf{x}(t) - \mathbf{x}_n(t) \quad \mathbf{u}_\delta(t) = \mathbf{u}(t) - \mathbf{u}_n(t) \quad \mathbf{y}_\delta(t) = \mathbf{y}(t) - \mathbf{y}_n(t)$$

- Now we rearrange our big equation to use these:

$$\dot{\mathbf{x}}(t) - \mathbf{f}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right] = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right]\left[\mathbf{x}(t)-\mathbf{x}_n(t)\right] + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right]\left[\mathbf{u}(t)-\mathbf{u}_n(t)\right]$$
$$\mathbf{y}(t) - \mathbf{h}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right] = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right]\left[\mathbf{x}(t)-\mathbf{x}_n(t)\right] + \frac{\partial \mathbf{h}}{\partial \mathbf{u}}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right]\left[\mathbf{u}(t)-\mathbf{u}_n(t)\right]$$

$$\dot{\mathbf{x}}(t) - \dot{\mathbf{x}}_n(t) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\left[\mathbf{x}\ \ (t),\mathbf{u}_n(t),t\right]\left[\mathbf{x}(t)-\mathbf{x}_n(t)\right] + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right]\left[\mathbf{u}(t)-\mathbf{u}_n(t)\right]$$
$$\mathbf{y}(t) - \mathbf{y}_n(t) = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right]\left[\mathbf{x}(t)-\mathbf{x}_n(t)\right] + \frac{\partial \mathbf{h}}{\partial \mathbf{u}}\left[\mathbf{x}_n(t),\mathbf{u}_n(t),t\right]\left[\mathbf{u}(t)-\mathbf{u}_n(t)\right]$$

$$\dot{\mathbf{x}}_\delta(t) = \dot{\mathbf{x}}(t) - \dot{\mathbf{x}}_n(t) \approx \mathbf{A}(t)\mathbf{x}_\delta(t) + \mathbf{B}(t)\mathbf{u}_\delta(t)$$
$$\mathbf{y}_\delta(t) = \mathbf{y}(t) - \mathbf{y}_n(t) \approx \mathbf{C}(t)\mathbf{x}_\delta(t) + \mathbf{D}(t)\mathbf{u}_\delta(t)$$

$$\dot{\mathbf{x}}_\delta(t) = \dot{\mathbf{x}}(t) - \dot{\mathbf{x}}_n(t) \approx \mathbf{A}(t)\mathbf{x}_\delta(t) + \mathbf{B}(t)\mathbf{u}_\delta(t)$$
$$\mathbf{y}_\delta(t) = \mathbf{y}(t) - \mathbf{y}_n(t) \approx \mathbf{C}(t)\mathbf{x}_\delta(t) + \mathbf{D}(t)\mathbf{u}_\delta(t)$$

- In many cases the non-linear functions, **f** and **h**, will be time-invariant

$$\dot{\mathbf{x}}(t) = f\left[\mathbf{x}(t), \mathbf{u}(t)\right]$$
$$\mathbf{y}(t) = h\left[\mathbf{x}(t), \mathbf{u}(t)\right]$$

- In this case, the matrices A, B, C, and D will be constant and the linearized system will be LTI.
- Another potential simplification occurs if the nominal trajectory just represents a constant equilibrium condition $x_n(t)=x_n$ for a constant nominal input $u_n(t)=u_n$. In this case, the derivative is zero:

$$0 = f\left[\mathbf{x}_n, \mathbf{u}_n\right]$$

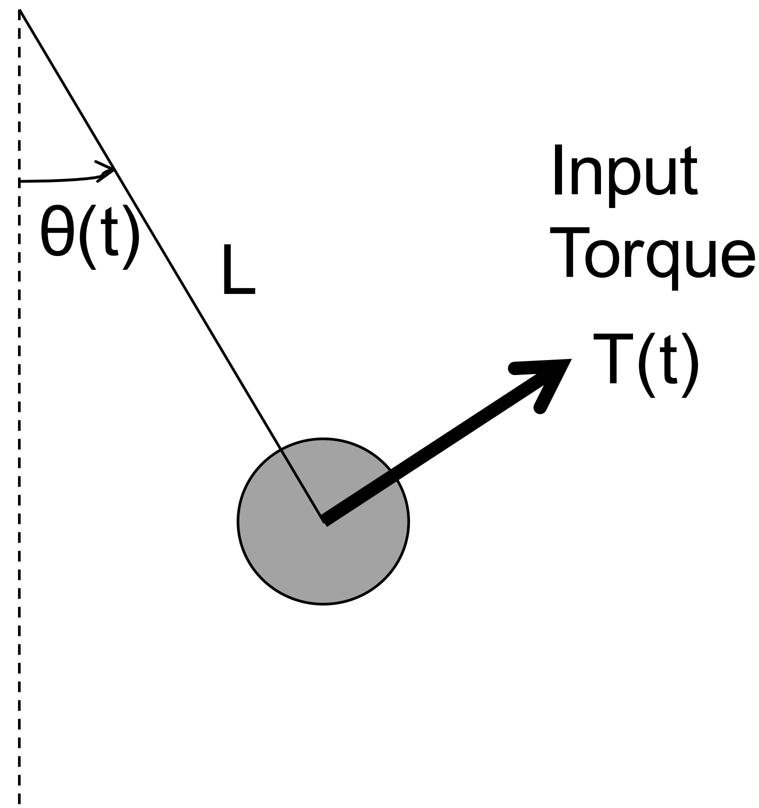
Linearization Example

- The motion of a pendulum on a taut string of length L is described by the following:

$$mL^2\ddot{\theta}(t) + mgL\sin(\theta(t)) = T(t)$$

- The use of sin(θ) makes this equation non-linear. We will linearize, but we first need to define our state variables. Assuming that θ(t) is the output:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$$



$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix} \qquad mL^2\ddot{\theta}(t) + mgL\sin(\theta(t)) = T(t)$$

- Lets try and write in SS form and see how far we get:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{g}{L}\sin(\theta(t)) + \frac{1}{mL^2}T(t) \end{aligned}$$

- Non-linear! But lets keep going with the output equation:

$$y(t) = \theta(t) = [1\ 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- The output equation is linear, so we only need to linearize the state equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t),t]$$
$$\mathbf{y}(t) = \mathbf{h}[\mathbf{x}(t),\mathbf{u}(t),t] \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{g}{L}\sin(\theta(t)) + \frac{1}{mL^2}T(t) \end{aligned}$$

- We will choose to linearize about a nominal input of $u_n(t) = 0$ and $x_n(t) = 0$. So our approximation will be good only around the stable equilibrium position (i.e. when θ is small). Here is the first-order Taylor expansion again, defined w.r.t. $x_n(t)$:

$$\dot{\mathbf{x}}(t) - \dot{\mathbf{x}}_n(t) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}[\mathbf{x}_n(t),\mathbf{u}_n(t),t][\mathbf{x}(t)-\mathbf{x}_n(t)] + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}[\mathbf{x}_n(t),\mathbf{u}_n(t),t][\mathbf{u}(t)-\mathbf{u}_n(t)]$$

- Since the nominal input and value are zero, we have:

$$\dot{\mathbf{x}}(t) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}[0,0,t][\mathbf{x}(t)] + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}[0,0,t][\mathbf{u}(t)]$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(0,0,t) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} \qquad \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(0,0,t) = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix}$$

- Finally, the linearized state equation!

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix} u(t)$$