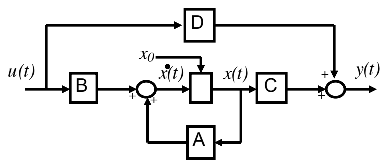

ENGI 7825: Control Systems II

The State-Space Representation: Part 1

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Adapted from the notes of
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What to Expect

- Math-heavy, implementation-light
  - Background: Linear algebra, differential equations, stuff from 5821
- Assignments: Mostly paper and pen, some Matlab
- Practical Assignments: Using V-REP robot simulator
  - The purpose is to motivate the techniques discussed and illustrate how they might be useful. Example problems drawn from mobile robotics.
- Mid-term tests and final: Based on core material presented in class and practiced on the assignments

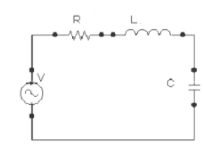
- Note: This course is not a hands-on introduction to industrial controls. It focuses instead on control systems theory.
  - (But remember that theory is the stuff that is harder to learn after graduating)

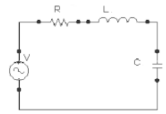
Why a Second Controls Course?

- Classical control techniques (ENGI 5821) only applicable to single-input, single-output (SISO) systems:
  - A heating system for one room
  - Orientation of a single joint on a robot
- Many real-world systems have multiple inputs and multiple outputs (MIMO)
  - A heating system for two rooms that accounts for heat transfer between them:
    - Input/Output: 2 temperatures
  - Simple two-wheeled mobile robot:
    - Input: 2 wheel speeds
    - Output: translational and rotational velocity
- Also, classical control techniques are usually based on the assumption of zero initial conditions; Lets review…

Classical Control

- The classical (i.e. frequency-domain) approach: converting a system's differential equation to a **transfer function**, thus generating a mathematical model of the system that relates the input to the output
- e.g. RLC circuit with *v(t)* as input and $v_c(t)$ as output





input: v(t)
output: $v_c(t)$

KVL: $$Ri(t)+L\frac{di(t)}{dt}+v_c(t)=v(t)$$

We need to relate the input *v(t)* to the output, $v_c(t)$; Replace *i(t)*:

$$i(t)=C\frac{dv_c(t)}{dt}$$

$$RC\frac{dv_C(t)}{dt}+LC\frac{d^2v_C(t)}{dt^2}+v_C(t)=v(t)$$

$$LC\frac{d^2v_C(t)}{dt^2}+RC\frac{dv_C(t)}{dt}+v_C(t)=v(t)$$

$$LC\frac{d^2v_C(t)}{dt^2}+RC\frac{dv_C(t)}{dt}+v_C(t)=v(t)$$

- Apply Laplace Transform using the following theorems:

Theorem
$\mathcal{L}\{\frac{df}{dt}\} = sF(s) - f(0-)$
$\mathcal{L}\{\frac{d^2f}{dt^2}\} = s^2F(s) - sf(0-) - f'(0-)$
$\mathcal{L}\{\frac{d^nf}{dt^n}\} = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0-)$

- If the initial conditions are all zero then we get

$$LCs^2V_C(s)+RCsV_C(s)+V_C(s)=V(s)$$

- If the initial conditions are non-zero we get

$$LCs^2V_C(s)+RCsV_C(s)+V_C(s)-LCsv_c(0)-LCv'_c(0)-RCv_c(0)=V(s)$$

- Zero initial conditions:

$$LCs^2V_C(s)+RCsV_C(s)+V_C(s)=V(s) \Rightarrow \frac{V_C(s)}{V(s)}=\frac{1/LC}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

- This is a transfer function (output / input)
- With non-zero initial conditions, there is no way to get a transfer function:

$$LCs^2V_C(s)+RCsV_C(s)+V_C(s)-LCsv_c(0)-LCv'_c(0)-RCv_c(0)=V(s)$$

$$\Rightarrow \frac{V_C(s)}{V(s)}=?$$

Advantages of State-Space Controls

- State-Space Controls is a method for modeling, analyzing and designing a wide range of systems, such as:
  - Nonzero initial condition systems
    - Classical restricted to zero initial conditions
  - Multiple-input, multiple-output systems (MIMO)
    - Classical restricted to single-input, single-output system (SISO)
- State-space has other advantages that we will see later on. For example:
  - Natural representation for nonlinear systems, time-varying systems (variable mass systems, …)

System State

- The state of a system is the information needed in addition to the input to determine the output
- A **memoryless** system's output is determined solely by the input
  - e.g. A single resistor with voltage as input and current as output
- The output of a **causal** system depends on past and current inputs. The impact of past inputs is represented by the state.
  - All physically realized systems are causal
- The output of a **non-causal** system depends on past, current, and future inputs! No real-world physical systems have this ability to anticipate future inputs
  - Non-causal systems can be simulated and non-causal filtering can be applied on previously collected data

Notation

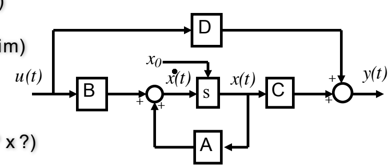
- We would like to describe general MIMO systems, so the inputs, outputs, and states will all be vectors
- input vector: **u(t)**
  - Assume we have control over the input
- output vector: **y(t)**
  - The output is any single variable or multiple variables that we can measure
- state vector: **x(t)**
  - Other variables representing the state of the system (may or may not be measureable)

State-Space Representation

- The state-space representation for an LTI system has the general form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \qquad \underbrace{\mathbf{x}(t_0) = \mathbf{x}_0}_{\text{Initial conditions}}$$

where:
x(t): state vector (n-dimensional)
y(t): output vector (p-dim)
u(t): input or control vector (m-dim)
A: dynamic matrix (? x ?)
B: input matrix (? x ?)
C: output matrix (? x ?)
D: feedforward (direct) matrix (? x ?)



Definitions

- **System state**: minimum information needed in order to completely determine the output of a system from a given moment, provided the input is known from that moment
- **System variable**: any variable that responds to an input or initial conditions in a system
- **State variables**: the smallest set of linearly independent system variables such that the values of the set members at time t_0 along with known inputs completely determine the value of all system variables for all $t \geq t_0$
- **State vector**: vector whose elements are the state variables
- **State space**: n-dimensional space whose axes represent the state variables
- **State equations**: set of n simultaneous, first order differential equations with n variables, where the n variables to be solved are the state variables
- **Output equation**: algebraic expression of the output variables of a system as linear combinations of the state variables and the inputs

Procedure for Selecting State Variables

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

- Consider the energy-storage elements in the system (capacitors and inductors in electrical syst.; masses and springs in mechanical syst.)
- For electrical systems you would then write the derivative equation for each energy-storage element

	Voltage-current	Current-voltage
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$
Capacit.	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$

- So for inductors, choose current as the state variable, for capacitors choose voltage
- Another perspective is to consider the relevant quantities in the energy equations: (inductor E = ½ L i^2, capacitor: E = ½ C v^2)

What do you do after selecting the state variables?

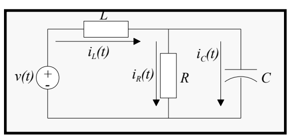
- Solve for the derivative of each state variable as a linear combination of system variables and the input
- Stack these scalar equations to form the state and output equations which operate on vectors
- The following is for a system with n = 4, p = 2 (outputs), m = 2 (inputs)

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} ? & ? & ? & ?\\ ? & ? & ? & ?\\ ? & ? & ? & ?\\ ? & ? & ? & ?\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}?\\ ?\\ ?\\ ?\end{bmatrix}u(t) \qquad \begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix} ? & ? & ? & ?\\ ? & ? & ? & ?\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}?\\ ?\end{bmatrix}u(t)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

State-space introductory example

- Example 1: Find a state-space representation of the system shown in the figure if the output is the current through the resistor.



- **Step 1** Select the state variables: write the derivative equation for the **energy-storage elements** (L and C)…

$$C\frac{dv_c}{dt} = i_C \qquad L\frac{di_L}{dt} = v_L$$

and choose the differentiated quantities as the state variables (v_C, i_L)

$$C\frac{dv_c}{dt} = i_C \qquad L\frac{di_L}{dt} = v_L$$

- **Step 2** Write the right-hand sides of "step 1" and the output equations as linear combinations of the state variables and the input.

$$C\frac{dv_c}{dt} = i_C = f_1(v_C,\, i_L,\, v(t)) \quad \rightarrow \quad i_C = -i_R + i_L = -\tfrac{1}{R}v_C + i_L$$

$$L\frac{di_L}{dt} = v_L = f_2(v_C,\, i_L,\, v(t)) \quad \rightarrow \quad v_L = -v_C + v(t)$$

State eq

$$\frac{dv_c}{dt} = -\tfrac{1}{RC}v_C + \tfrac{1}{C}i_L$$
$$\frac{di_L}{dt} = -\tfrac{1}{L}v_C + \tfrac{1}{L}v(t)$$

Output eq $i_R = f_3(v_C, i_L, v(t)) \rightarrow i_R = \tfrac{1}{R}v_C$

$$\frac{dv_c}{dt} = -\tfrac{1}{RC} v_C + \tfrac{1}{C} i_L$$

$$\frac{di_L}{dt} = -\tfrac{1}{L} v_C + \tfrac{1}{L} v(t)$$

$$i_R = \tfrac{1}{R} v_C$$

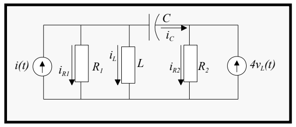
- **Step 4** Obtain the S-S representation in vector-matrix form

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \cdot \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v(t)$$

$$i_R = \begin{bmatrix} \frac{1}{R} & 0 \end{bmatrix} \cdot \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

Another electrical example

- Example 2: Obtain the SS equations for the next circuit, with input *i(t)* and output vector $y=[v_{R2}\ i_{R2}]^T$



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States Eq

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1 C\Delta} & \frac{1-4R_2}{C\Delta} \\ -\frac{1}{L\Delta} & \frac{R_2}{L\Delta} \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} -\frac{1-4R_2}{C\Delta} \\ -\frac{R_2}{L\Delta} \end{bmatrix} i(t)$$

Out Eq

$$\begin{bmatrix} v_{R2} \\ i_{R2} \end{bmatrix} = \begin{bmatrix} -(1+\frac{1}{\Delta}) & \frac{R_2}{\Delta} \\ \frac{-(1+\frac{1}{\Delta})}{R_2} & \frac{1}{\Delta} \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} -\frac{R_2}{\Delta} \\ -\frac{1}{\Delta} \end{bmatrix} i(t)$$

Where

$$\Delta = -\left((1-4R_2) + \frac{R_2}{R_1}\right)$$

Mechanical systems

- The procedure for mechanical systems is a bit different, but we will still assign a state variable for each energy-storage element.

TABLE 2.4 Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring ($x(t)$, $f(t)$, K)	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper ($x(t)$, $f(t)$, f_v)	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass ($x(t)$, M, $f(t)$)	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m(newton-seconds/meter), M = kg(kilograms = newton-seconds²/meter).

Energy storage elements

- A spring has potential energy and a moving mass has kinetic energy. A viscous damper is analogous to a resistor in that it does not store energy.

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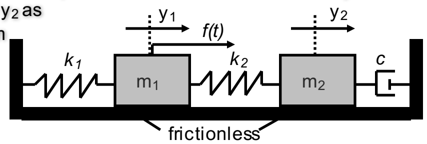
Shows x(t) as displacement but we'll use y(t) since that is the intended output in our examples

- A spring has potential energy ½ *k y(t)*² so we will use displacement, y(t) as a state variable
- A moving mass has kinetic energy ½ m$\dot{y}(t)^2$ so we will use the speed, $\dot{y}(t)$, as a state variable

A mechanical example

- Example 2: Find the state eqs for the translational mechanical system shown in figure, with y_1 and y_2 as outputs of the system and f(t) the input

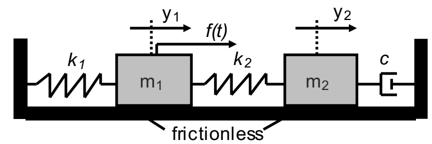


- What state variables should we define?
- For each mass we use the displacement (for the spring's potential energy) and speed (for the mass's kinetic energy)
- The following are all reasonable choices for state variables:

$$x_1(t)=y_1(t) \quad x_2(t)=y_2(t) \quad x_3(t)=\dot{y}_1(t) \quad x_4(t)=\dot{y}_2(t)$$

$$x_1(t)=y_1(t) \quad x_2(t)=\dot{y}_1(t) \quad x_3(t)=y_2(t) \quad x_4(t)=\dot{y}_2(t)$$

$$x_1(t)=y_1(t) \quad x_2(t)=y_2(t)-y_1(t) \quad x_3(t)=\dot{y}_1(t) \quad x_4(t)=\dot{y}_2(t)$$



We use the following state variables:

$$x_1(t)=y_1(t) \quad x_2(t)=y_2(t) \quad x_3(t)=\dot{y}_1(t) \quad x_4(t)=\dot{y}_2(t)$$

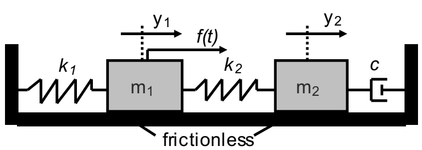
Consider the motion of each mass and apply Newton's second law: **m a = Σ F**.
First consider the forces on m_1:

$$m_1\ddot{y}_1(t)=-k_1y_1(t)-k_2y_1(t)+k_2y_2(t)+f(t)$$

$$m_1\ddot{y}_1(t)=-(k_1+k_2)y_1(t)+k_2y_2(t)+f(t)$$

Now consider the forces on m_2:

$$m_2\ddot{y}_2(t)=k_2y_1(t)-k_2y_2(t)-c\dot{y}_2(t)$$



$$x_1(t)=y_1(t) \quad x_2(t)=y_2(t) \quad x_3(t)=\dot{y}_1(t) \quad x_4(t)=\dot{y}_2(t)$$

We derived the following on the previous slide:

$$m_1\ddot{y}_1(t)=-(k_1+k_2)y_1(t)+k_2y_2(t)+f(t)$$

$$m_2\ddot{y}_2(t)=k_2y_1(t)-k_2y_2(t)-c\dot{y}_2(t)$$

Combining into matrix-vector form yields the following SS representation:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0\\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & -\frac{c}{m_2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \frac{1}{m_1}\\ 0\end{bmatrix}f(t) \qquad \begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}0\\ 0\end{bmatrix}f(t)$$