

ENGI 7825: Control Systems II

State-Space Fundamentals: Part 2: The Matrix Exponential

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Adapted from the notes of
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A “Matrixy” Integrating Factor

- ▶ Go back through the solution to the scalar system and see what operations are performed using the integrating factor:

- Multiplication on both sides
- Differentiation
- Multiplication on both sides of the “inverse” $e^{a(t-t_0)}$

- ▶ We need some kind of matrix entity that can do all of the above in the same way as $e^{-a(t-t_0)}$

- ▶ What is special about differentiation of exponential functions of the form e^{at} ?

$$\frac{d}{dt}e^{at} = ae^{at}$$

- ▶ We need something with the same property in order to solve an SS system for $x(t)$.

The Matrix Exponential

- ▶ The scalar exponential is defined by the infinite power series:

$$e^{at} = 1 + at + \frac{1}{2}a^2t^2 + \frac{1}{6}a^3t^3 + \dots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} a^k t^k$$

- ▶ We define the matrix exponential by replacing a with the matrix A :

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \dots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$

- ▶ Although the notation is surprising, e^{At} is really just a matrix, defined to behave in a similar way to the scalar exponential.

- ▶ Once again, e^{At} is just notation used to represent a power series. In general, the matrix exponential does not equal the matrix of scalar exponentials of the elements in the matrix A .

$$e^{At} \neq [e^{a_{ij}t}]$$

- ▶ Example 1: Consider the following 4x4 matrix: $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$

Lets obtain the first few terms of the power series:

$$A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad A^k = 0 \quad \forall k \geq 4$$

The power series contains only a finite number of nonzero terms:

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -t & 1 & 0 & 0 \\ \frac{1}{2}t^2 & -t & 1 & 0 \\ -\frac{1}{6}t^3 & \frac{1}{2}t^2 & -t & 1 \end{bmatrix} \quad \neq [e^{a_{ij}t}] = \begin{bmatrix} e^0 & 0 & 0 & 0 \\ e^{-t} & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^{-t} & 0 \end{bmatrix}$$

► Example 2: For a diagonal matrix, this equality is satisfied $e^{A^t} = [e^{a_{ij}^t}]$

Consider the diagonal nxn matrix A:

$$A = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_{n-1} & 0 \\ 0 & 0 & \dots & 0 & \lambda_n \end{bmatrix}$$

The power series contains an infinite number of terms:

$$A^k = \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 & 0 \\ 0 & \lambda_2^k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_{n-1}^k & 0 \\ 0 & 0 & \dots & 0 & \lambda_n^k \end{bmatrix}$$

Thus:

$$e^{A^t} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 & 0 \\ 0 & \lambda_2^k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_{n-1}^k & 0 \\ 0 & 0 & \dots & 0 & \lambda_n^k \end{bmatrix} t^k = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_1^k t^k & 0 & \dots & 0 & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_2^k t^k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_{n-1}^k t^k & 0 \\ 0 & 0 & \dots & 0 & \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_n^k t^k \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & e^{\lambda_{n-1} t} & 0 \\ 0 & 0 & \dots & 0 & e^{\lambda_n t} \end{bmatrix}$$

Properties of the matrix exponential

► For any real nxn matrix A, the matrix exponential e^{A^t} satisfies:

1. e^{A^t} is the unique matrix for which: $\frac{d}{dt} e^{A^t} = A e^{A^t}$ $e^{A^t} \Big|_{t=0} = I (nxn)$
2. For any t_1 and t_2 : $e^{A(t_1+t_2)} = e^{A t_1} e^{A t_2}$

As a consequence: $e^{A(0)} = e^{A(t-t)} = e^{A^t} e^{-A^t} = I$

Thus, e^{A^t} is invertible for all t, with the inverse: $[e^{A^t}]^{-1} = e^{-A^t}$

3. For all t, A and e^{A^t} commute with respect to matrix product $A e^{A^t} = e^{A^t} A$
4. For all t: $[e^{A^t}]^T = e^{A^T t}$
5. For any real nxn matrix B, $e^{(A+B)t} = e^{A^t} e^{B^t}$ for all t if and only if $AB=BA$
6. Finally, a useful property of the matrix exponential is that it can be reduced to a finite power series involving n scalar analytic functions $\alpha_i(t)$

$$e^{A^t} = \sum_{k=0}^{n-1} \alpha_k(t) A^k$$

State Equation Solution

► The solution of the state differential equation is found to be:

$$\mathbf{x}(t) = e^{A(t-t_0)} \mathbf{x}_0 + \int_{t_0}^t e^{A(t-\tau)} \mathbf{B} \mathbf{u}(\tau) d\tau$$

where the matrix exponential is needed:

$$e^{A(t-t_0)} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k (t-t_0)^k = I + A(t-t_0) + \frac{1}{2} A^2 (t-t_0)^2 + \frac{1}{6} A^3 (t-t_0)^3 + \dots$$

► The matrix exponential is sometimes referred to as the state-transition matrix and denoted by $\Phi(t, t_0)$:

$$\Phi(t, t_0) = e^{A(t-t_0)}$$

► Using this notation, the solution of the state differential equation can be written as:

$$\mathbf{x}(t) = \underbrace{\Phi(t, t_0) \mathbf{x}_0}_{\text{zero-input response: } \mathbf{x}_{zi}(t)} + \underbrace{\int_{t_0}^t \Phi(t, \tau) \mathbf{B} \mathbf{u}(\tau) d\tau}_{\mathbf{x}_{zs}(t): \text{ zero-state response}}$$

Output Equation Solution

► Having the solution for the complete state response, a solution for the complete output equation can be obtained as:

$$\mathbf{y}(t) = \underbrace{C e^{A(t-t_0)} \mathbf{x}_0}_{\text{zero-input output: } \mathbf{y}_{zi}(t)} + \underbrace{\int_{t_0}^t C e^{A(t-\tau)} \mathbf{B} \mathbf{u}(\tau) d\tau + \mathbf{D} \mathbf{u}(t)}_{\mathbf{y}_{zs}(t): \text{ zero-state output}}$$

► Any issues with these solutions for x(t) and y(t)?

► Yes! How can we evaluate $e^{A(t-t_0)}$ when its defined by an infinite series?

► Solution: We will look to the frequency domain for the solution for the zero-state response, which will lead to another way of evaluating $e^{A(t-t_0)}$.

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)} \mathbf{x}_0 + \int_{t_0}^t e^{\mathbf{A}(t-\tau)} \mathbf{B} \mathbf{u}(\tau) d\tau$$

- The solution to the unforced system ($\mathbf{u}=0$) is simply

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} \phi_{11}(t) & \cdots & \phi_{n1}(t) \\ \phi_{21}(t) & \cdots & \phi_{n2}(t) \\ \vdots & & \vdots \\ \phi_{n1}(t) & \cdots & \phi_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix}$$

therefore, the term $\phi_{ij}(t)$ can be interpreted (and determined) as the response of the i^{th} state variable due to an initial condition on the j^{th} state variable when there are zero initial conditions on all other states

Frequency Domain Solution

- The solution for $\mathbf{x}(t)$ proposed so far is based on the mysterious matrix exponential. Ignoring the weirdness of $e^{\mathbf{A}t}$, the bigger problem is how to compute it.
- We will pursue an alternate solution in the frequency domain which will actually lead to another way of computing $e^{\mathbf{A}t}$.

$$L[\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)]$$

$$L[\dot{\mathbf{x}}(t)] = L[\mathbf{A}\mathbf{x}(t)] + L[\mathbf{B}\mathbf{u}(t)]$$

$$s\mathbf{X}(s) - \mathbf{x}_0 = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{X}(s) = \underbrace{(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{x}_0}_{\text{zero-input response: } \mathbf{X}_{zi}(s)} + \underbrace{(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}\mathbf{U}(s)}_{\mathbf{X}_{zs}(s): \text{zero-state response}}$$

$$\mathbf{X}(s) = \underbrace{(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{x}_0}_{\text{zero-input response: } \mathbf{X}_{zi}(s)} + \underbrace{(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}\mathbf{U}(s)}_{\mathbf{X}_{zs}(s): \text{zero-state response}}$$

zero-input response: $\mathbf{X}_{zi}(s)$ $\mathbf{X}_{zs}(s)$: zero-state response

- This frequency-domain solution bears a strong resemblance to our previous solution:

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)} \mathbf{x}_0 + \int_{t_0}^t e^{\mathbf{A}(t-\tau)} \mathbf{B} \mathbf{u}(\tau) d\tau$$

- In fact, we can conclude that the matrix exponential, $e^{\mathbf{A}(t-t_0)}$ and $(s\mathbf{I} - \mathbf{A})^{-1}$ form a Laplace transform pair:

$$L[e^{\mathbf{A}(t-t_0)}] = [s\mathbf{I} - \mathbf{A}]^{-1} \iff e^{\mathbf{A}(t-t_0)} = L^{-1}[s\mathbf{I} - \mathbf{A}]^{-1}$$

- We should also finish the frequency domain solution by obtaining the output:

$$\begin{aligned} Y(s) &= C\mathbf{X}(s) + D\mathbf{U}(s) \\ &= C(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{x}_0 + [C(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + D]\mathbf{U}(s) \end{aligned}$$

Impulse Response

- We can divide the output into zero-input and zero-state components:

$$Y(s) = \underbrace{C(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{x}_0}_{\text{zero-input output: } Y_{zi}(s)} + \underbrace{[C(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + D]\mathbf{U}(s)}_{Y_{zs}(s): \text{zero-state output}}$$

- Consider just $Y_{zs}(s)$ and recall that multiplication in the frequency domain is convolution in the time domain.

$$Y_{zs}(s) = [C(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + D]\mathbf{U}(s)$$

An impulse in the time-domain is just a 1 in the frequency domain. So the impulse response in the frequency domain (i.e. the transfer function) is easily found:

$$H(s) = C(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + D$$

State-Space Fundamentals: exercise

- Solve the following linear second-order ordinary differential eq:

$$\ddot{y}(t) + 7\dot{y}(t) + 12y(t) = u(t)$$

Consider the input $u(t)$ is a step of magnitude 3

and the initial conditions: $\dot{y}(0) = 0.05$ $y(0) = 0.10$

First choose state variables in controller canonical form:

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} = \dot{x}_1 \\ \dot{x}_2 &= \dot{y} = u_s - 12x_1 - 7x_2 \end{aligned}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0.10 \\ 0.05 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u(t)$$

$$A = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix}$$

Powers of A are not nulls, thus, obtaining the state transition matrix as a power series is impractical

$$A = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix}$$

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s)$$

- The matrix $(sI - A)^{-1}$ is required

$$sI - A = \begin{bmatrix} s & -1 \\ 12 & s+7 \end{bmatrix}$$

$$\det(sI - A) = |sI - A| = s^2 + 7s + 12$$

$$(sI - A)^{-1} = \frac{1}{s^2 + 7s + 12} \begin{bmatrix} s+7 & 1 \\ -12 & s \end{bmatrix}$$

- Look back on your old linear algebra notes for a quick refresher on matrix inversion. Another source is Williams and Lawrence, Appendix A.

$$(sI - A)^{-1} = \frac{1}{s^2 + 7s + 12} \begin{bmatrix} s+7 & 1 \\ -12 & s \end{bmatrix}$$

- Thus, from $X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s)$

$$X(s) = \frac{1}{s^2 + 7s + 12} \begin{bmatrix} s+7 & 1 \\ -12 & s \end{bmatrix} \begin{bmatrix} 0.10 \\ 0.05 \end{bmatrix} + \frac{1}{s^2 + 7s + 12} \begin{bmatrix} s+7 & 1 \\ -12 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{3}{s}$$

$$= \frac{1}{s^2 + 7s + 12} \begin{bmatrix} 0.1s + 0.75 + \frac{3}{s} \\ 0.05s + 1.8 \end{bmatrix} = \frac{\begin{bmatrix} 0.1s^2 + 0.75s + 3 \\ s(s+3)(s+4) \end{bmatrix}}{\begin{bmatrix} 0.05s + 1.8 \\ (s+3)(s+4) \end{bmatrix}} = \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}$$

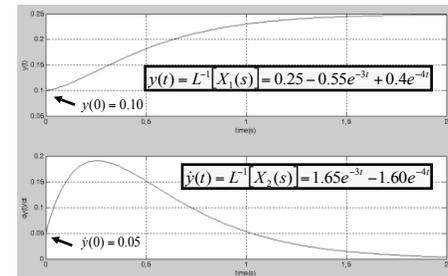
$$X_1(s) = \frac{0.1s^2 + 0.75s + 3}{s(s+3)(s+4)} = \frac{0.25}{s} - \frac{0.55}{s+3} + \frac{0.4}{s+4}$$

$$y(t) = L^{-1}\{X_1(s)\} = 0.25 - 0.55e^{-3t} + 0.4e^{-4t}$$

$$X_2(s) = \frac{0.05s + 1.8}{(s+3)(s+4)} = \frac{1.65}{s+3} - \frac{1.60}{s+4}$$

$$\dot{y}(t) = L^{-1}\{X_2(s)\} = 1.65e^{-3t} - 1.60e^{-4t}$$

Take derivative to confirm



- The general expression of the state-transition matrix can be obtained from

$$e^{At} = L^{-1}[(sI - A)^{-1}] = L^{-1} \left[\frac{1}{s^2 + 7s + 12} \begin{bmatrix} s+7 & 1 \\ -12 & s \end{bmatrix} \right] = \begin{bmatrix} 4e^{-3t} - 3e^{-4t} & e^{-3t} - e^{-4t} \\ -12e^{-3t} + 12e^{-4t} & -3e^{-3t} + 4e^{-4t} \end{bmatrix}$$

- But what if $t_0 \neq 0$? Then t is just replaced with $t - t_0$,

$$e^{A(t-t_0)} = L^{-1}[(sI - A)^{-1}] = \begin{bmatrix} 4e^{-3(t-t_0)} - 3e^{-4(t-t_0)} & e^{-3(t-t_0)} - e^{-4(t-t_0)} \\ -12e^{-3(t-t_0)} + 12e^{-4(t-t_0)} & -3e^{-3(t-t_0)} + 4e^{-4(t-t_0)} \end{bmatrix}$$