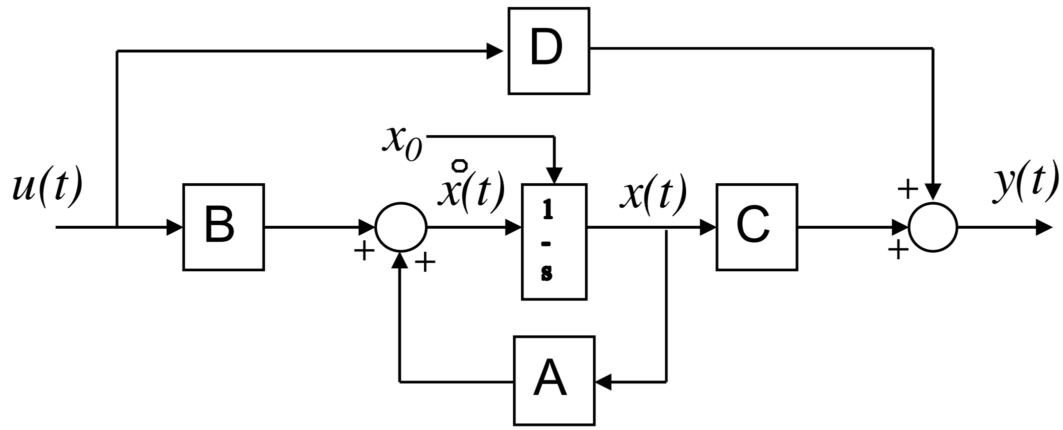

ENGI 7825: Control Systems II

State-Space Fundamentals:
Part 2: The Matrix Exponential

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Adapted from the notes of
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A "Matrixy" Integrating Factor

- Go back through the solution to the scalar system and see what operations are performed using the integrating factor:
  - Multiplication on both sides
  - Differentiation
  - Multiplication on both sides of the "inverse" $e^{a(t-t_0)}$
- We need some kind of matrix entity that can do all of the above in the same way as $e^{-a(t-t0)}$
- What is special about differentiation of exponential functions of the form e^{at}?

$$\frac{d}{dt}e^{at} = ae^{at}$$

- We need something with the same property in order to solve an SS system for x(t).

The Matrix Exponential

- The scalar exponential is defined by the infinite power series:

$$e^{at} = 1 + at + \tfrac{1}{2}a^2t^2 + \tfrac{1}{6}a^3t^3 + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} a^k t^k$$

- We define the matrix exponential by replacing *a* with the matrix *A*:

$$e^{At} = I + At + \tfrac{1}{2}A^2t^2 + \tfrac{1}{6}A^3t^3 + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$

- Although the notation is surprising, e^{At} is really just a matrix, defined to behave in a similar way to the scalar exponential.

- Once again, $e^{\mathbf{A}t}$ is just notation used to represent a power series. In general, the matrix exponential does not equal the matrix of scalar exponentials of the elements in the matrix A.

$$e^{\mathbf{A}t} \neq [e^{a_{ij}t}]$$

- Example 1: Consider the following 4x4 matrix: $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$

  Lets obtain the first few terms of the power series:

$$A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad A^k = 0 \quad \forall k \geq 4$$

  The power series contains only a finite number of nonzero terms:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2t^2 + \frac{1}{6}\mathbf{A}^3t^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -t & 1 & 0 & 0 \\ \frac{1}{2}t^2 & -t & 1 & 0 \\ -\frac{1}{6}t^3 & \frac{1}{2}t^2 & -t & 1 \end{bmatrix} \quad \neq \left[e^{a_{ij}t}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ e^{-t} & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^{-t} & 0 \end{bmatrix}$$

- Example 2: For a diagonal matrix, this equality **is** satisfied: $e^{\mathbf{A}t} = [e^{a_{ij}t}]$

Consider the diagonal nxn matrix A: $A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix}$

The power series contains an infinite number of terms: $A^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2^k & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1}^k & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n^k \end{bmatrix}$

Thus:

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2^k & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1}^k & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n^k \end{bmatrix} t^k = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_1^k t^k & 0 & \cdots & 0 & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_2^k t^k & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_{n-1}^k t^k & 0 \\ 0 & 0 & \cdots & 0 & \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_n^k t^k \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_{n-1} t} & 0 \\ 0 & 0 & \cdots & 0 & e^{\lambda_n t} \end{bmatrix}$$

Properties of the matrix exponential

- For any real nxn matrix **A**, the matrix exponential $e^{\mathbf{A}t}$ satisfies:

1. $e^{\mathbf{A}t}$ is the unique matrix for which: $\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} \qquad e^{\mathbf{A}t}\Big|_{t=0} = \mathbf{I}(nxn)$
2. For any t_1 and t_2: $e^{\mathbf{A}(t_1+t_2)} = e^{\mathbf{A}t_1}e^{\mathbf{A}t_2}$

   As a consequence: $e^{\mathbf{A}(0)} = e^{\mathbf{A}(t-t)} = e^{\mathbf{A}t}e^{-\mathbf{A}t} = \mathbf{I}$

   Thus, $e^{\mathbf{A}t}$ is invertible for all t, with the inverse: $\left[e^{\mathbf{A}t}\right]^{-1} = e^{-\mathbf{A}t}$

3. For all t, A and $e^{\mathbf{A}t}$ commute with respect to matrix product: $\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$
4. For all t: $\left[e^{\mathbf{A}t}\right]^{T} = e^{\mathbf{A}^{T}t}$
5. For any real nxn matrix B, $e^{(\mathbf{A}+\mathbf{B})t} = e^{\mathbf{A}t}e^{\mathbf{B}t}$ for all t if and only if **AB**=**BA**
6. Finally, a useful property of the matrix exponential is that it can be reduced to a finite power series involving n scalar analytic functions $\alpha_j(t)$

$$e^{\mathbf{A}t} = \sum_{k=0}^{n-1}\alpha_k(t)\mathbf{A}^{k}$$

State Equation Solution

- The solution of the state differential equation is found to be:

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x_0} + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{Bu}(\tau)\,d\tau$$

where the matrix exponential is needed:

$$e^{\mathbf{A}(t-t_0)} = \sum_{k=0}^{\infty}\frac{1}{k!}\mathbf{A}^k(t-t_0)^k = \mathbf{I} + \mathbf{A}(t-t_0) + \frac{1}{2}\mathbf{A}^2(t-t_0)^2 + \frac{1}{6}\mathbf{A}^3(t-t_0)^3 + \ldots$$

- The matrix exponential is sometimes referred to as the state-transition matrix and denoted by $\phi(t, t_0)$:

$$\Phi(t,t_0) = e^{\mathbf{A}(t-t_0)}$$

- Using this notation, the solution of the state differential equation can be written as:

$$\mathbf{x}(t) = \underbrace{\Phi(t,t_0)\mathbf{x_0}}_{\text{zero-input response: } x_{zi}(t)} + \underbrace{\int_{t_0}^{t} \Phi(t,\tau)\mathbf{Bu}(\tau)\,d\tau}_{x_{zs}(t)\text{: zero-state response}}$$

zero-input response: $x_{zi}(t)$

$x_{zs}(t)$: zero-state response

Output Equation Solution

- Having the solution for the complete state response, a solution for the complete output equation can be obtained as:

$$\mathbf{y}(t) = \underbrace{\mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{x_0}}_{\text{zero-input output: } y_{zi}(t)} + \underbrace{\int_{t_0}^{t} \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau + \mathbf{D}\mathbf{u}(t)}_{y_{zs}(t)\text{: zero-state output}}$$

- Any issues with these solutions for x(t) and y(t)?
- Yes! How can we evaluate $e^{A(t-t0)}$ when its defined by an infinite series?
- Solution: We will look to the frequency domain for the solution for the zero-state response, which will lead to another way of evaluating $e^{A(t-t0)}$.

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x_0} + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

- The solution to the unforced system (u=0) is simply

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} \phi_{11}(t) & \cdots & \phi_{n1}(t) \\ \phi_{21}(t) & \cdots & \phi_{n2}(t) \\ \vdots & & \vdots \\ \phi_{n1}(t) & \cdots & \phi_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix}$$

therefore, the term $\phi_{ij}(t)$ can be interpreted (and determined) as the response of the i^{th} state variable due to an initial condition on the j^{th} state variable when there are zero initial conditions on all other states

Frequency Domain Solution

- The solution for x(t) proposed so far is based on the mysterious matrix exponential. Ignoring the weirdness of e^{At}, the bigger problem is how to compute it.
- We will pursue an alternate solution in the frequency domain which will actually lead to another way of computing e^{At}.

$$L\left[\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\right]$$

$$L\left[\dot{\mathbf{x}}(t)\right] = L\left[\mathbf{A}\mathbf{x}(t)\right] + L\left[\mathbf{B}\mathbf{u}(t)\right]$$

$$s\mathbf{X}(s) - \mathbf{x}_0 = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{X}(s) = \underbrace{(s\mathbf{I} - \mathbf{A})^{-1}}_{\text{zero-input response: } X_{zi}(s)}\mathbf{x}_0 + \underbrace{(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)}_{X_{zs}(s)\text{: zero-state response}}$$

zero-input response: $X_{zi}(s)$

$X_{zs}(s)$: zero-state response

$$\mathbf{X}(s) = \underbrace{(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x_0}}_{\text{zero-input response: } X_{zi}(s)} + \underbrace{(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)}_{X_{zs}(s)\text{: zero-state response}}$$

- This frequency-domain solution bears a strong resemblance to our previous solution:

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x_0} + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

- In fact, we can conclude that the matrix exponential, $e^{A(t-t_0)}$ and $(sI - A)^{-1}$ form a Laplace transform pair:

$$L\left[e^{A(t-t_0)}\right] = \left[s\mathbf{I} - \mathbf{A}\right]^{-1} \quad \longleftrightarrow \quad e^{A(t-t_0)} = L^{-1}\left[s\mathbf{I} - \mathbf{A}\right]^{-1}$$

- We should also finish the frequency domain solution by obtaining the output:

$$\begin{aligned} Y(s) &= CX(s) + DU(s) \\ &= C(sI - A)^{-1}x_0 + [C(sI - A)^{-1}B + D]U(s) \end{aligned}$$

Impulse Response

- We can divide the output into zero-input and zero-state components:

$$Y(s) = \underbrace{C(sI - A)^{-1}x_0}_{\text{zero-input output: } Y_{zi}(s)} + \underbrace{[C(sI - A)^{-1}B + D]U(s)}_{Y_{zs}(s)\text{: zero-state output}}$$

- Consider just $Y_{zs}(s)$ and recall that multiplication in the frequency domain is convolution in the time domain.

$$Y_{zs}(s) = [C(sI - A)^{-1}B + D]U(s)$$

An impulse in the time-domain is just a 1 in the frequency domain. So the impulse response in the frequency domain (i.e. the transfer function) is easily found:

$$H(s) = C(sI - A)^{-1}B + D$$

State-Space Fundamentals: exercise

- Solve the following linear second-order ordinary differential eq:

$$\ddot{y}(t) + 7\dot{y}(t) + 12y(t) = u(t)$$

Consider the input u(t) is a step of magnitude 3

and the initial conditions: $\dot{y}(0) = 0.05 \quad y(0) = 0.10$

First choose state variables in controller canonical form:

$$x_1 = y$$
$$x_2 = \dot{y} = \dot{x}_1$$
$$\dot{x}_2 = \ddot{y} = u_s - 12x_1 - 7x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix}$$

B

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0.10 \\ 0.05 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \cdot u(t)$$

C D

Powers of A are not nulls, thus, obtaining the state transition matrix as a power series is impractical

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix}$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x_0} + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{BU}(s)$$

- The matrix ($s\mathbf{I}$ - $\mathbf{A}$)$^{-1}$ is required:

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 \\ 12 & s+7 \end{bmatrix}$$

$$\det(s\mathbf{I} - \mathbf{A}) = |s\mathbf{I} - \mathbf{A}| = s^2 + 7s + 12$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + 7s + 12}\begin{bmatrix} s+7 & 1 \\ -12 & s \end{bmatrix}$$

- Look back on your old linear algebra notes for a quick refresher on matrix inversion. Another source is Williams and Lawrence, Appendix A.

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{s^2+7s+12}\begin{bmatrix} s+7 & 1 \\ -12 & s \end{bmatrix}$$

- Thus, from $\mathbf{X}(s) = (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x_0} + (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{BU}(s)$

$$\mathbf{X}(s) = \frac{1}{s^2+7s+12}\begin{bmatrix} s+7 & 1 \\ -12 & s \end{bmatrix}\begin{bmatrix} 0.10 \\ 0.05 \end{bmatrix} + \frac{1}{s^2+7s+12}\begin{bmatrix} s+7 & 1 \\ -12 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{3}{s} =$$

$$= \frac{1}{s^2+7s+12}\begin{bmatrix} 0.1s+0.75+\frac{3}{s} \\ 0.05s+1.8 \end{bmatrix} = \begin{bmatrix} \dfrac{0.1s^2+0.75s+3}{s(s+3)(s+4)} \\ \dfrac{0.05s+1.8}{(s+3)(s+4)} \end{bmatrix} = \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}$$

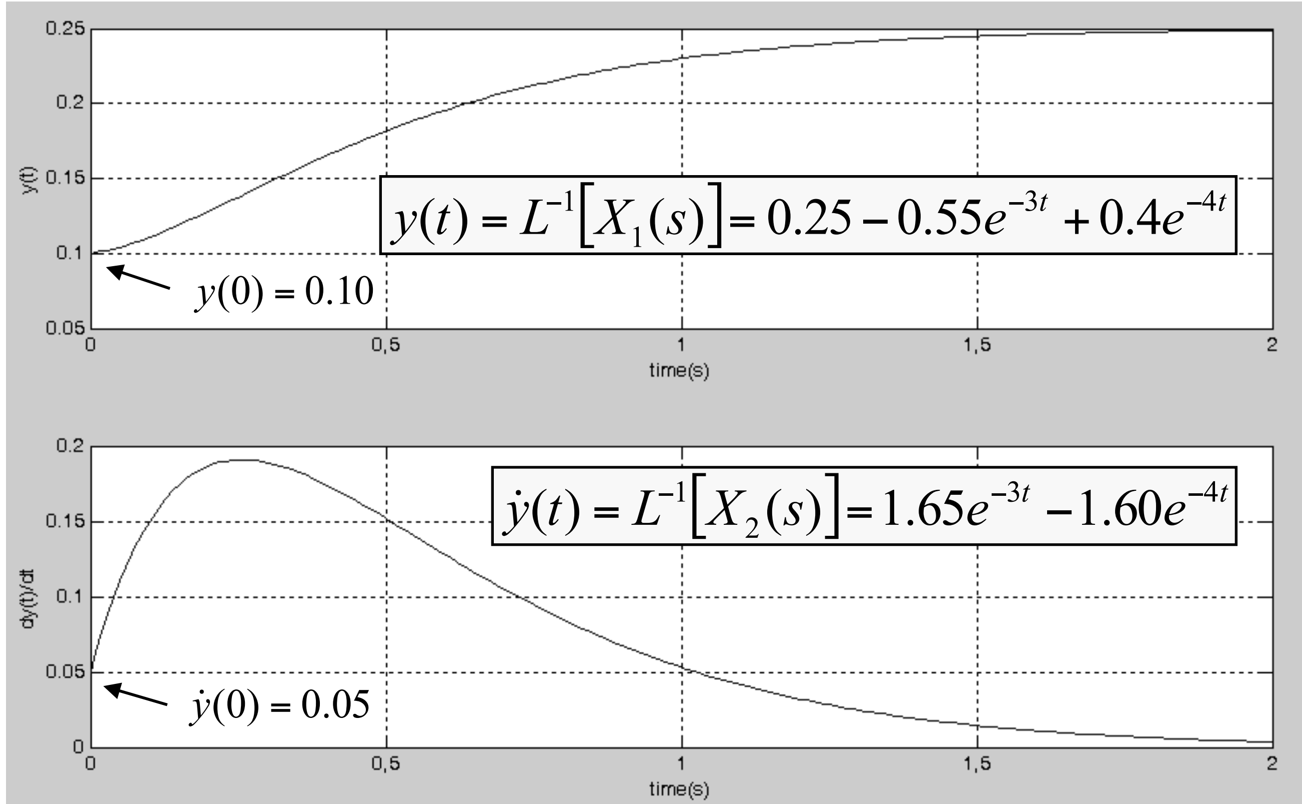
$$X_1(s) = \frac{0.1s^2+0.75s+3}{s(s+3)(s+4)} = \frac{0.25}{s} - \frac{0.55}{s+3} + \frac{0.4}{s+4}$$

$$y(t) = L^{-1}\left[X_1(s)\right] = 0.25 - 0.55e^{-3t} + 0.4e^{-4t}$$

Take derivative to confirm

$$X_2(s) = \frac{0.05s+1.8}{(s+3)(s+4)} = \frac{1.65}{s+3} - \frac{1.60}{s+4}$$

$$\dot{y}(t) = L^{-1}\left[X_2(s)\right] = 1.65e^{-3t} - 1.60e^{-4t}$$



- The general expression of the state-transition matrix can be obtained from

$$e^{At} = L^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right] = L^{-1}\left[\frac{1}{s^2+7s+12}\begin{bmatrix} s+7 & 1 \\ -12 & s \end{bmatrix}\right] = \begin{bmatrix} 4e^{-3t}-3e^{-4t} & e^{-3t}-e^{-4t} \\ -12e^{-3t}+12e^{-4t} & -3e^{-3t}+4e^{-4t} \end{bmatrix}$$

- But what if $t_0 \neq 0$? Then t is just replaced with $t - t_0$,

$$e^{A(t-t_0)} = L^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right] = \begin{bmatrix} 4e^{-3(t-t_0)}-3e^{-4(t-t_0)} & e^{-3(t-t_0)}-e^{-4(t-t_0)} \\ -12e^{-3(t-t_0)}+12e^{-4(t-t_0)} & -3e^{-3(t-t_0)}+4e^{-4(t-t_0)} \end{bmatrix}$$