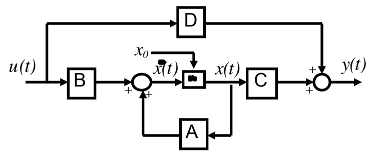

ENGI 7825: Control Systems II

State-Space Fundamentals: Part 1

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Adapted from the notes of
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Introduction

- The basic mathematical model for an LTI system consists of the

  state equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad \mathbf{x}(t_0) = \mathbf{x}_0$

  and the **output equation** $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$

- Rather than dive into the full solution for this vector-matrix equation, we will start by deriving the solution to the

$$\dot{x}(t) = ax(t) + bu(t) \qquad x(t_0) = x_0$$
$$y(t) = cx(t) + du(t)$$

- If x(t) is known then the output equation follows directly from the state equation. Therefore we focus exclusively on the state equation.
- Begin by re-writing in the standard form for a first-order DE:

$$\dot{x}(t) - ax(t) = bu(t)$$

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- A standard solution technique for first-order DE's is to multiply both sides by the following integrating factor:

$$e^{-a(t-t_0)}$$

- This yields the following:

$$e^{-a(t-t_0)}\dot{x}(t) - e^{-a(t-t_0)}ax(t) = e^{-a(t-t_0)}bu(t)$$

- That didn't seem to help! Try taking the derivative of the integrating factor, multiplied by x(t):

$$\frac{d}{dt}\left[e^{-a(t-t_0)}x(t)\right] = e^{-a(t-t_0)}\dot{x}(t) - e^{-a(t-t_0)}ax(t)$$

$$e^{-a(t-t_0)}\dot{x}(t) - e^{-a(t-t_0)}ax(t) = e^{-a(t-t_0)}bu(t)$$
$$\frac{d}{dt}\left[e^{-a(t-t_0)}x(t)\right] = e^{-a(t-t_0)}\dot{x}(t) - e^{-a(t-t_0)}ax(t)$$

- So we can replace the LHS of the top equation with the LHS of the bottom:

$$\frac{d}{dt}\left[e^{-a(t-t_0)}x(t)\right] = e^{-a(t-t_0)}bu(t)$$

- We can now integrate both sides to try and expose x(t):

$$\int_{t_0}^{t} \frac{d}{d\tau}\left[e^{-a(\tau-t_0)}x(\tau)\right] d\tau = \int_{t_0}^{t} e^{-a(\tau-t_0)}bu(\tau)d\tau$$

- Prior to integrating we changed the variable from t to ¿ to avoid confusion with the upper limit of integration

$$\int_{t_0}^{t} \frac{d}{d\tau}\left[e^{-a(\tau-t_0)}x(\tau)\right]d\tau = \int_{t_0}^{t} e^{-a(\tau-t_0)}bu(\tau)d\tau$$

- The following is a corollary of the fundamental theorem of calculus:

$$\int_a^b \frac{d}{dt}f(t)dt = f(b) - f(a)$$

- Applying this and a little algebra yields our final solution!

$$\begin{aligned}
e^{-a(t-t_0)}x(t) - e^{-a(t_0-t_0)}x(t_0) &= \int_{t_0}^{t} e^{-a(\tau-t_0)}bu(\tau)d\tau \\
e^{-a(t-t_0)}x(t) - x(t_0) &= \int_{t_0}^{t} e^{-a(\tau-t_0)}bu(\tau)d\tau \\
x(t) &= e^{a(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{a(t-\tau)}bu(\tau)d\tau
\end{aligned}$$

$$x(t) = e^{a(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{a(t-\tau)}bu(\tau)d\tau$$

- To refer to the initial conditions, we may use $x(t_0)$ or just x_0:

$$x(t) = \underbrace{e^{a(t-t_0)}x_0}_{\substack{\text{zero-input response} \\ \text{“natural response”}}} + \underbrace{\int_{t_0}^{t} e^{a(t-\tau)}bu(\tau)d\tau}_{\substack{\text{zero-state response} \\ \text{“forced response”}}}$$

- The first part is known as the **zero-input response** (or **natural response**) and represents the response with no input.
- The second part is the **zero-state response (or forced response)** and represents the response of the system to the input, assuming that the initial state was zero.

$$x(t) = e^{a(t-t_0)}x_0 + \int_{t_0}^{t} e^{a(t-\tau)}bu(\tau)d\tau$$

$$y(t) = cx(t) + du(t)$$

- We can substitute x(t) directly into the output equation to obtain y(t):

$$y(t) = ce^{a(t-t_0)}x_0 + \int_{t_0}^{t} ce^{a(t-\tau)}bu(\tau)d\tau + du(t)$$

$$y(t) = ce^{a(t-t_0)}x_0 + \int_{t_0}^{t} ce^{a(t-\tau)}bu(\tau)d\tau + du(t)$$

- We often like to characterize a system by its impulse response. This is obtained by setting $u(t) = \delta(t)$ under zero initial conditions, $x_0 = 0$ (the zero vector) at $t_0=0^-$:

$$\begin{aligned}
h(t) &= \int_{0^-}^{t} ce^{a(t-\tau)}b\delta(\tau)d\tau + d\delta(t) \\
&= ce^{a(t)}b + d\delta(t)
\end{aligned}$$

- We used the sifting property of the impulse to pluck out the value of the integrated function at 0. We can then use the impulse response to get the zero-state response output for any u(t):

$$\begin{aligned}
\int_{0^-}^{t} ce^{a(t-\tau)}bu(\tau)\,d\tau + du(t) &= \int_{0^-}^{t}[ce^{a(t-\tau)}b + d\delta(t-\tau)]u(\tau)\,d\tau \\
&= \int_{0^-}^{t} h(t-\tau)u(\tau)\,d\tau \\
&= h(t) * u(t)
\end{aligned}$$

From scalar solution to vector solution

So we obtained the solution to the scalar system,

$$\dot{x}(t) = ax(t) + bu(t) \qquad x(t_0) = x_0$$
$$y(t) = cx(t) + du(t)$$

using the integrating factor,

$$e^{-a(t-t_0)}$$

resulting in,

$$x(t) = \underbrace{e^{a(t-t_0)}x_0}_{x_{zi}(t)} + \underbrace{\int_{t_0}^{t} e^{a(t-\tau)}bu(\tau)d\tau}_{x_{zs}(t)}$$

The solution to the vector system,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad \mathbf{x}(t_0) = \mathbf{x_0}$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

can be obtained in the exact same way using the integrating factor,

$$e^{-A(t-t_0)}$$

resulting in,

$$x(t) = \underbrace{e^{A(t-t_0)}x_0}_{x_{zi}(t)} + \underbrace{\int_{t_0}^{t} e^{A(t-\tau)}bu(\tau)d\tau}_{x_{zs}(t)}$$

But what the hell is this?

$$e^{-A(t-t_0)}$$