ENGI 7825: Linear Algebra Review
Finding Eigenvalues and Diagonalization

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June 24, 2016

Eigenvalues for Triangular Matrices

The eigenvalues for a triangular matrix are the entries of the diagonal.

Reason: Suppose A is an upper triangular matrix and a_{kk} is one of the diagonal entries. Then $B = A - a_{kk}I$ is an upper triangular matrix with a 0 in the k^{th} diagonal spot.

So B is in echelon form with (at least) x_k as a free variable. The existence of a free variable means the null space is not empty. The null space of B is exactly the eigenspace for the eigenvalue a_{kk}.

Defining Equation for Eigenvectors and Eigenvalues

$$A\vec{v} = \lambda\vec{v}$$

How do you find the eigenvalues for a matrix in general?

Recall: Let $B = A - \lambda I$ be an $n \times n$ matrix. Then

$$\text{Nul}\, B = \vec{0} \iff B \text{ invertible} \iff \det B \neq 0.$$

Translation: let $\lambda \in \mathbb{R}$ and $B = A - \lambda I$.

$$\lambda \text{ is eigenvalue for } A \iff \text{Nul}\, B \neq \vec{0} \iff \det B = 0.$$

In other words, the eigenvalues of A are exactly the values λ for which $\det(A - \lambda I) = 0$!

Since $\det(A - \lambda I)$ is a polynomial, this is a familiar problem.

Example: Find all eigenvalues for the matrix $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$.

Answer: Find values of λ for which

$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix} = 0$$

In other words, find roots of the quadratic equation $\lambda^2 - 5\lambda + 4 = 0$.

We've known since high school how to do this!

$$\lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1) = 0 \implies \lambda = 1 \text{ or } 4.$$

Check: Both $A - 4I = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$ and $A - I = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$ clearly have non-empty null spaces.

Example: Find all eigenvalues and corresponding eigenvectors for the matrix $A = \begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix}$.

Solution:

Step 1: Find the eigenvalues. That is, find all the solutions to the characteristic equation for A:

$$\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & 6 \\ 3 & -2-\lambda \end{bmatrix} = 0$$

Calculate:

$$\det \begin{bmatrix} 5-\lambda & 6 \\ 3 & -2-\lambda \end{bmatrix} = (5-\lambda)(-2-\lambda) - 18 = \lambda^2 - 3\lambda - 28$$

Note that $\lambda^2 - 3\lambda - 28$ is called the characteristic polynomial of A.

$$\lambda^2 - 3\lambda - 28 = (\lambda - 7)(\lambda + 4) = 0 \iff \lambda = 7 \text{ or } \lambda = -4$$

Hence eigenvalues are $\lambda = 7$ and $\lambda = -4$.

Step 2: Calculate eigenvectors for eigenvalue $\lambda = 7$:

This is the null space of

$$A - 7 \cdot I = \begin{bmatrix} 5-7 & 6 \\ 3 & -2-7 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ 3 & -9 \end{bmatrix} \rightarrow_{reduce} \begin{bmatrix} -2 & 6 \\ 0 & 0 \end{bmatrix}$$

Solution: Eigenspace is all multiples of vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Check:

$$\begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 21 \\ 7 \end{bmatrix} = 7 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

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Step 3: Calculate eigenvectors for eigenvalue $\lambda = -4$:

This is the null space of

$$A - (-4) \cdot I = \begin{bmatrix} 5+4 & 6 \\ 3 & -2+4 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 3 & 2 \end{bmatrix} \to_{reduce} \begin{bmatrix} 9 & 6 \\ 0 & 0 \end{bmatrix}$$

Solution: Eigenspace is all multiples of vector $\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$

Check:

$$\begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{8}{3} \\ -4 \end{bmatrix} = -4 \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

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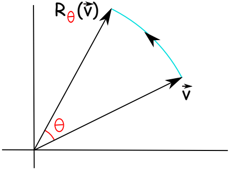
Example: Find the eigenvalues for the matrix $A = \begin{bmatrix} 3/2 & -13 \\ 1/4 & -3/2 \end{bmatrix}$.

$\det(A - \lambda I) = \lambda^2 + 1$.

There are two complex eigenvalues $\pm i$. We solve for the eigenspace in the usual way, although the matrices $A - iI$ and $A + iI$ will now be complex.

Final Example: Rotation: matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.



Here is what we said before:

> *"In general, a rotated vector is moved to a new direction, so for most values of θ there will be no eigenvectors! (Hence no eigenvalues either)."*

However, this is not exactly true! The question is whether the eigenvalues are complex. Lets go ahead and determine them...

$$\begin{aligned} R_\theta - \lambda I &= \begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix} \\ \det(R_\theta - \lambda I) &= (\cos\theta - \lambda)^2 + \sin^2\theta \\ &= \cos^2\theta - 2\cos\theta\lambda + \lambda^2 + \sin^2\theta \\ &= \lambda^2 - 2\cos\theta\lambda + 1 \end{aligned}$$

Setting this to 0 and solving for λ we get,

$$\begin{aligned} \lambda &= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} \\ \lambda &= \cos\theta \pm \sqrt{cos^2\theta - 1} \end{aligned}$$

Try some particular values of θ. In general, the eigenvalues will be complex whenever $\cos^2\theta < 1$. So we only get real eigenvalues when θ is an integer multiple of π.

Why should we care about eigenvalues and eigenvectors?

One reason:

Suppose you plan to apply a matrix linear transformation A to a vector $\vec{v}$ lots of times: $AAAAA...A\vec{v} = A^{bazillion}\vec{v}$.

If $\vec{v}$ is an eigenvector and λ is its eigenvalue, then $A^{bazillion}\vec{v} = \lambda^{bazillion}\vec{v}$. So it's easy to calculate.

Example (from above): Let $A = \begin{bmatrix} 5 & 6 \\ 3 & -2 \end{bmatrix}$. What is $A^{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$?

Solution: Since $A\begin{bmatrix} 3 \\ 1 \end{bmatrix} = 7\begin{bmatrix} 3 \\ 1 \end{bmatrix}$,

$$AAAAAAAAAA\begin{bmatrix} 3 \\ 1 \end{bmatrix} = 7 \cdot 7 \cdot \ldots \cdot 7\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 7^{10} \\ 7^{10} \end{bmatrix}$$

Pushing further - can often get complete formula for $A^{bazillion}$.

Example: Have shown $A\begin{bmatrix}3\\1\end{bmatrix} = 7\begin{bmatrix}3\\1\end{bmatrix}$ and $A\begin{bmatrix}-\frac{2}{3}\\1\end{bmatrix} = -4\begin{bmatrix}-\frac{2}{3}\\1\end{bmatrix}$.

Combining these vector equations into a matrix equation:

$$A\begin{bmatrix}3 & -\frac{2}{3}\\1 & 1\end{bmatrix} = \begin{bmatrix}7\cdot 3 & -4\cdot -\frac{2}{3}\\7\cdot 1 & -4\cdot 1\end{bmatrix} = \begin{bmatrix}3 & -\frac{2}{3}\\1 & 1\end{bmatrix}\begin{bmatrix}7 & 0\\0 & -4\end{bmatrix}$$

Diagonalization

We can generalize the left and right sides of the above equation:

$$AV = VD$$

where V is composed of the eigenvectors of A arranged as columns and D is a diagonal matrix with the corresponding eigenvalues along the diagonal. Its often useful to rewrite this equation as the following diagonalization of A:

$$A = VDV^{-1}$$

Returning to our example, we can compute A^{1000} using the relation $A = VDV^{-1}$.

If we denote $\begin{bmatrix} 3 & -\frac{2}{3} \\ 1 & 1 \end{bmatrix} = V$ we have

$$AV = V \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} \implies A = V \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} V^{-1} \implies$$

$$A^{1000} = V \begin{bmatrix} 7^{1000} & 0 \\ 0 & (-4)^{1000} \end{bmatrix} V^{-1}$$

But why would anyone ever need to calculate A^{1000}?

Example Suppose w_0 is the number of wolves in the forest at time $t = 0$,

r_0 is the number of rabbits. Consider the following simple biological model:

- Wolves make more wolves. If there are rabbits to eat, they make even more wolves!
- Similarly rabbits make (lots) more rabbits. But rabbits also get eaten by wolves.

So, if in year zero there are w_0 wolves & r_0 rabbits, then next year, the number of wolves w_1 and rabbits r_1 might be given by:

$$w_1 = \frac{w_0}{2} + \frac{2r_0}{5} \qquad (1)$$

$$r_1 = -\frac{w_0}{10} + \frac{3r_0}{2} \qquad (2)$$

The constants $\frac{1}{2}$, $\frac{2}{5}$, $\frac{-1}{10}$, and $\frac{3}{2}$ would come from biological experiments.

In other words, $\begin{bmatrix} w_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ -\frac{1}{10} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} w_0 \\ r_0 \end{bmatrix}$.

If the same model applies the following year, then after that second year there are w_2 wolves & r_2 rabbits where

$$\begin{bmatrix} w_2 \\ r_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ -\frac{1}{10} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} w_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ -\frac{1}{10} & \frac{3}{2} \end{bmatrix} \left(\begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ -\frac{1}{10} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} w_0 \\ r_0 \end{bmatrix} \right)$$

In other words,

$$\begin{bmatrix} w_2 \\ r_2 \end{bmatrix} = A^2 \begin{bmatrix} w_0 \\ r_0 \end{bmatrix}$$

In general, after k years we have w_k wolves & r_k rabbits:

$$\begin{bmatrix} w_k \\ r_k \end{bmatrix} = A^k \begin{bmatrix} w_0 \\ r_0 \end{bmatrix}$$

Characteristic equation of $\begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ -\frac{1}{10} & \frac{3}{2} \end{bmatrix}$ is $\lambda^2 - 2\lambda + \frac{79}{100}$.

Roots of $\lambda^2 - 2\lambda + \frac{79}{100} = 0$ can be found via the quadratic formula:

$$\lambda = \frac{2 \pm \sqrt{4 - \frac{79}{25}}}{2} = 1 \pm \sqrt{\frac{21}{100}}$$

Now find eigenvectors for each, and assemble into the matrix V.

$$A^{1000} = V \begin{bmatrix} (1 + \sqrt{\frac{21}{100}})^{1000} & 0 \\ 0 & (1 - \sqrt{\frac{21}{100}})^{1000} \end{bmatrix} V^{-1}$$

Multiply by the initial condition vector to get the number of wolves and rabbits after 1000 years!

Definition

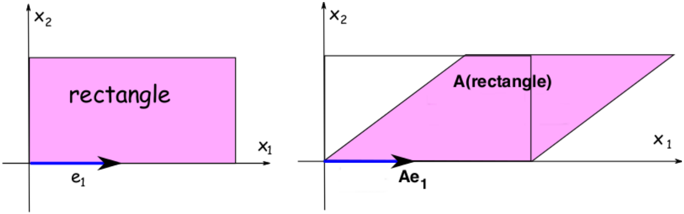
If A is a square matrix and there is an invertible matrix P so that $A = PBP^{-1}$ then A and B are similar.
Two matrices, A and B, which are similar will share the same eigenvalues.
In the diagonalization $A = VDV^{-1}$ the matrices A and D are similar by design.

Question: When can you diagonalize A?

Theorem

Answer: Can diagonalize A if and only if A has n linearly independent eigenvectors.

non-Example: Recall the shear matrix which yielded the following transformation:



$A = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$ has repeated eigenvalues $\lambda_1 = \lambda_2 = 1$ which both have the same eigenspace (multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$).

This matrix is not diagonalizable because diagonalizability requires n linearly independent eigenvectors, which is equivalent to having n distinct eigenvalues.