ENGI 7825: Linear Algebra Review
Introduction to Eigenvectors and Eigenvalues

Adapted from Notes Developed by Martin Scharlemann

June 24, 2016

The effect of a multiplication by matrix A on a vector $\vec{v}$ may be hard to describe. However, that transformation may have no effect on some vectors other than changing their length. In this case, the vector is an eigenvector for that transformation and the change in length is the eigenvalue.

Definition

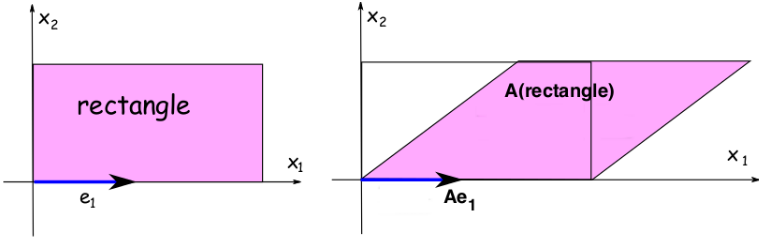
Suppose for some $\lambda \in \mathbb{R}$ and $\vec{v} \neq 0 \in \mathbb{V}$,

$$A\vec{v} = \lambda\vec{v}$$

Then $\vec{v}$ is an eigenvector for A and λ is its eigenvalue.

Thinking geometrically, $\vec{v}$ has the special property that its direction is the same, but its size is scaled by $\lambda \in \mathbb{R}$.

Example 1: Consider this shear transformation:



Its matrix is $A = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$.

Notice the horizontal vector $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ stays exactly the same.

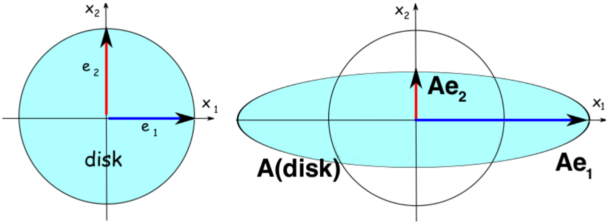
Since

$$A\vec{e_1} = \vec{e_1} = 1 \cdot \vec{e_1}$$

the vector $\vec{e_1}$ is an eigenvector of A with eigenvalue 1.

Actually all non-zero vectors along the x_1-axis (i. e. all horizontal vectors) are eigenvectors with eigenvalue 1.

Example 2: Scaling: matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.



Notice the horizontal $\vec{e}_1$ is doubled and the vertical $\vec{e}_2$ is halved.

Since

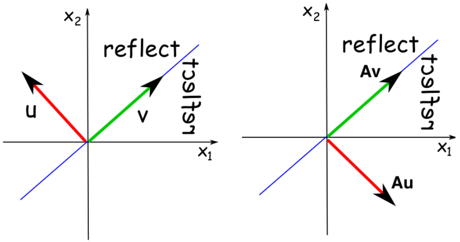
$$A\vec{e}_1 = 2 \cdot \vec{e}_1; \qquad and \qquad A\vec{e}_2 = \frac{1}{2} \cdot \vec{e}_2$$

$\vec{e}_1$ is eigenvector of A with eigenvalue 2;

$\vec{e}_2$ is eigenvector of A with eigenvalue $\frac{1}{2}$.

Notice that $\vec{e}_1 + \vec{e}_2$ is not an eigenvector: $A(\vec{e}_1 + \vec{e}_2) = 2\vec{e}_1 + \frac{1}{2}\vec{e}_2$.

Example 3: Reflection: matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.



Notice $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ stays same and $\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ points opposite.

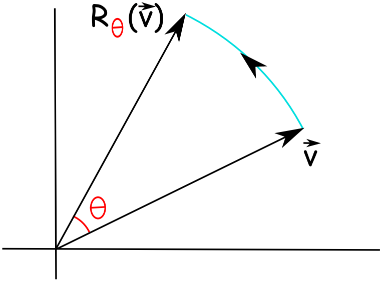
Since

$$A\vec{v} = 1 \cdot \vec{v}; \qquad and \qquad A\vec{u} = -1 \cdot \vec{u}$$

$\vec{v}$ is eigenvector for L with eigenvalue $+1$;

$\vec{u}$ is eigenvector for L with eigenvalue -1.

Example 4: Rotation: matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.



In general a rotated vector is moved to a new direction, so for most values of θ there will be no eigenvectors! (Hence no eigenvalues either). Are there some values of θ for which there are eigenvalues?

Suppose you know an eigenvalue λ of an $n \times n$ matrix A.

Theorem

The set E_λ of eigenvectors for A with eigenvalue λ (together with $\vec{0}$) is a subspace of $\mathbb{R}^n$. (Called the eigenspace for λ.)

This is just another way of saying that there may be many (in fact a whole subspace) eigenvectors associated with a particular eigenvalue.

Suppose we are given a matrix A and an eigenvalue λ. We want to find all $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$:

$$A\vec{v} = \lambda\vec{v} \iff A\vec{v} = \lambda(I\vec{v}) = (\lambda I)\vec{v} \iff (A - \lambda I)\vec{v} = \vec{0}$$

where I is the $n \times n$ identity matrix.

Hence for matrix A, eigenspace E_λ with eigenvalue λ is the nullspace of $A - \lambda I$.

$$E_\lambda = \mathrm{Nul}(A - \lambda I).$$

And we know how to find the basis of a nullspace.

Example: Your boss tells you that 4 is an eigenvalue for the matrix $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$. She wants you to find all eigenvectors.

Answer:

$$A - 4I = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$$

We must fine $E_4 = \text{Nul} \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$.

This matrix in echelon form is $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$. Any solution to the equation

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

has the form $x_1 = x_2$, so the eigenvectors are all multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Example: Your boss tells you that 3 is an eigenvalue for the matrix $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$. She wants you to find all eigenvectors for this eigenvalue, in other words the eigenspace E_3.

Answer:

$$A - 3I = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{bmatrix}$$

This matrix in echelon form is $B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

So... find a basis for the nullspace of B

Finding a basis for Nul B:

Any solution to the equation

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$

has the form

$$x_1 + x_2 - x_3 = 0 \implies x_1 = -x_2 + x_3,$$

so the eigenspace of A with eigenvalue 3 is spanned by the vectors

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Green shows that vectors linearly independent, hence basis for E_3.

Check: Are $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ eigenvectors of eigenvalue 3 for A?

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Yes! Both are eigenvectors because they satisfy $A\vec{v} = \lambda\vec{v}$.