Digital Control: Part 2

ENGI 7825: Control Systems II

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Mapping the s-plane onto the z-plane

- We're almost ready to design a controller for a DT system, however we will have to consider where we would like to position the poles
- We generally understand how to position desirable poles in the s-plane
  - Although this does remains somewhat of a "black art" as there are various arbitrary choices and rules-of-thumb at play
- If we understand how to position poles in the z-plane we can do direct digital design. Alternatively, we can position poles in the s-plane and then find out where they lie in the z-plane.

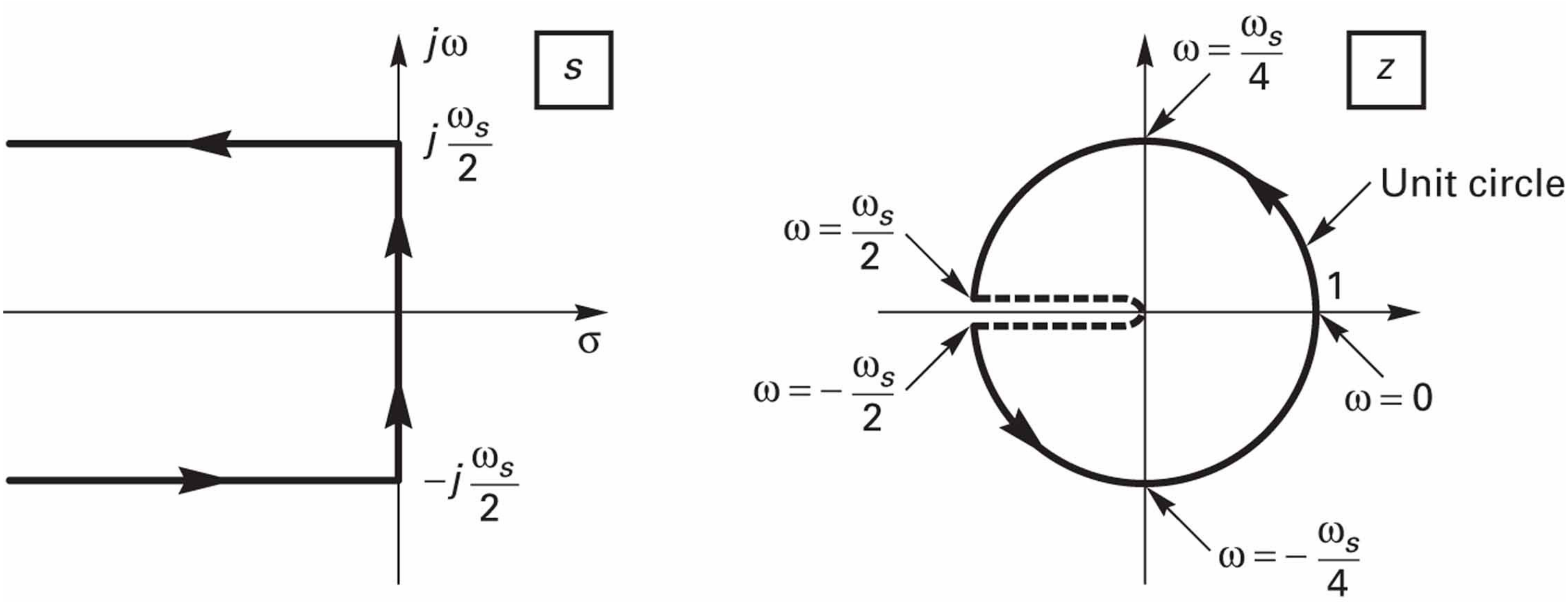
- We have already seen that poles in the s-plane and z-plane are related by

$$z = e^{sT}$$

- We'll consider particular mappings from parts of the s-plane. We have already seen that the j! axis corresponds to the unit circle in the z-plane. In the following, $s = \sigma + j\omega$ and $\omega = 0$.

$$z = e^{sT} = e^{\sigma T} e^{j\omega T} = e^{j\omega T} = 1\angle\omega T$$

- Fundamentally, there is a limitation on the signal frequency that can be represented by the z-transform. That limit is $\omega = \omega_s / 2$ where $\omega_s = 2\pi / T$.

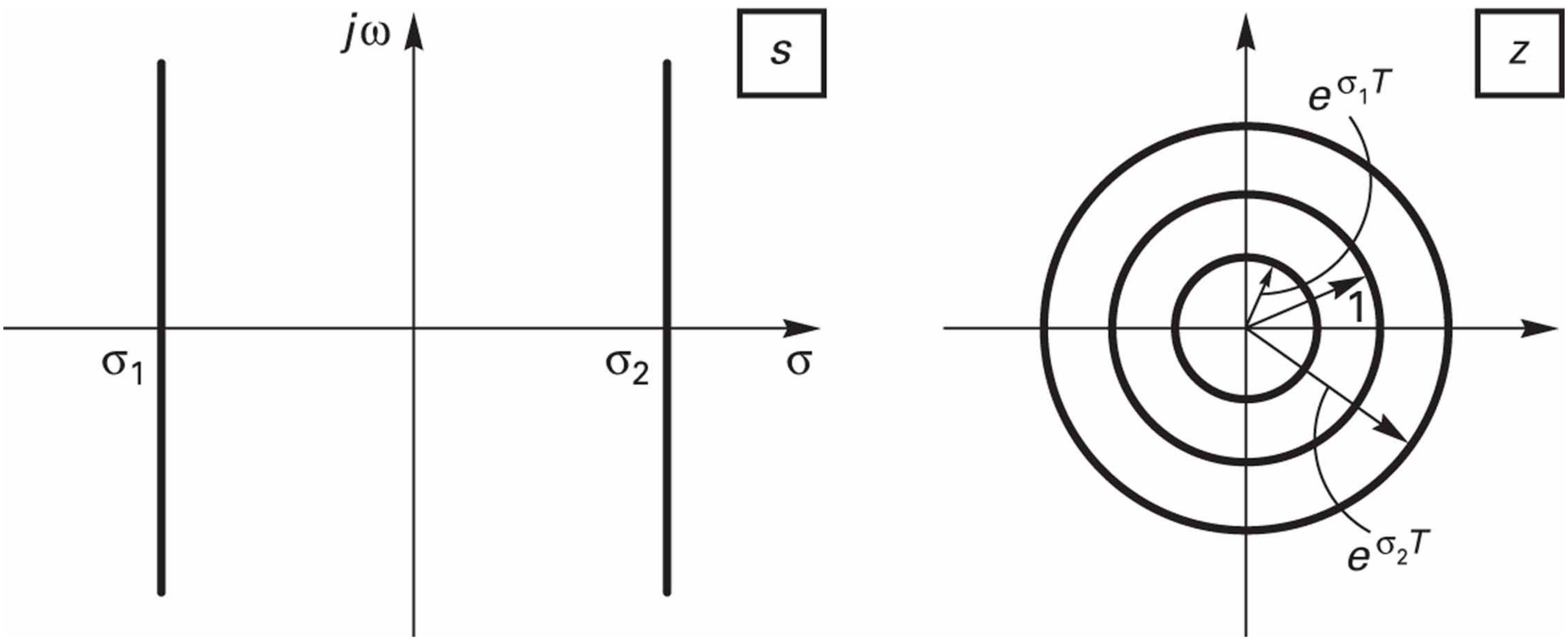


- That portion of the j! axis which lies in the range [$-j\omega_s/2$, $j\omega_s/2$] maps onto the unit circle.
- So poles on the unit circle in the z-plane correspond to pure sinusoids and therefore signify a <u>marginally stable system</u>.
- As we have already seen, poles inside the unit circle correspond to exponentially decaying sinusoids. If all poles lie within the unit circle then we have <u>asymptotic stability</u>. Poles outside the unit circle correspond to exponentially growing sinusoids, and therefore <u>instability</u>.

For s = $\sigma + j\omega$ if σ is held constant (lets say we set it to a value of σ_1) and ω is allowed to vary we get

$$z = e^{\sigma_1 T} e^{j\omega T} = e^{\sigma_1 T} \angle \omega T$$

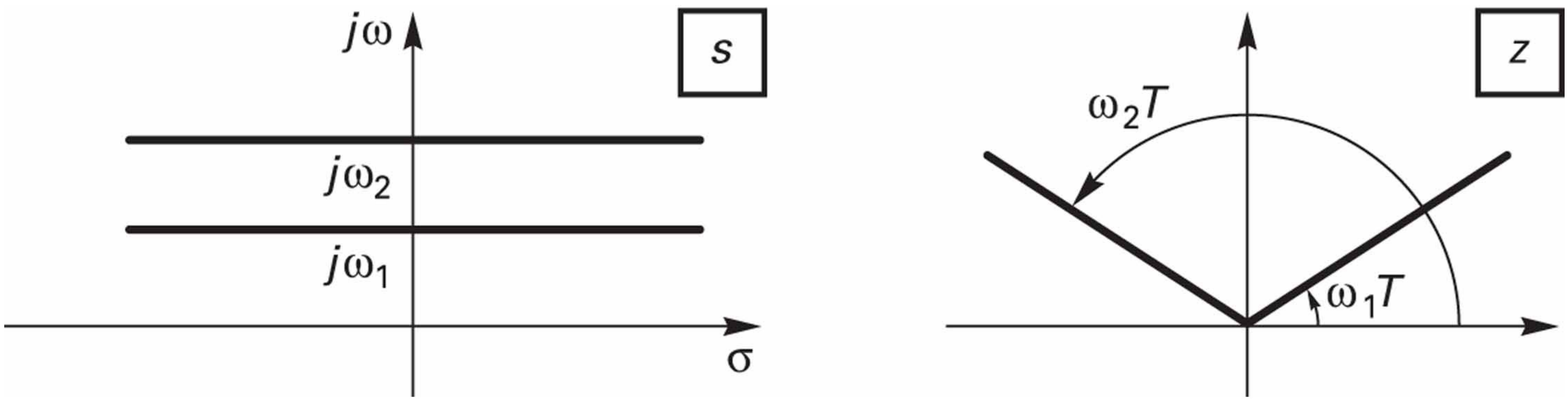
This corresponds to vertical lines in the s-plane and circles in the z-plane (including the unit circle).



What if we do the opposite? That is, for s = σ + jω we hold ω constant (at ω_1) if σ is allowed to vary allowed to vary we get

$$z = e^{\sigma T} e^{j\omega_1 T} = e^{\sigma T} \angle(\omega_1 T)$$

This corresponds to horizontal lines in the s-plane and rays emanating from the origin in the z-plane.



Lets consider pairs of poles located at $s = \sigma \pm j\omega$. We know that such a pole pair corresponds to a term of the form $ke^{\sigma t}\cos(\omega t + \psi)$. We can also define this pair of poles in polar coordinates as $(r, \pm\theta)$ as below:

$$z = e^{sT}\big|_{s=\sigma\pm j\omega} = e^{\sigma T}e^{\pm j\omega T} = e^{\sigma T}\angle \pm\omega T = r\angle \pm\theta$$

In particular we would like to position the poles of a second-order system which have the following locations:

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Now translate to the z-plane:

$$z_{1,2} = e^{sT}\big|_{s=s_{1,2}} = e^{-\zeta\omega_n T}e^{\pm j\omega_n T\sqrt{1-\zeta^2}} = re^{\pm j\theta}$$

$$\text{where } r = e^{-\zeta\omega_n T} \text{ and } \theta = \omega_n T\sqrt{1-\zeta^2}$$

We can then solve for the relationship between $(r, \pm\theta)$ and (ζ, ω_n):

$$\zeta = \frac{-\ln r}{\sqrt{\ln^2 r + \theta^2}} \qquad \omega_n = \frac{1}{T}\sqrt{\ln^2 r + \theta^2}$$

The relationships between z-plane pole locations and (ζ, ω_n) is somewhat complex, geometrically:

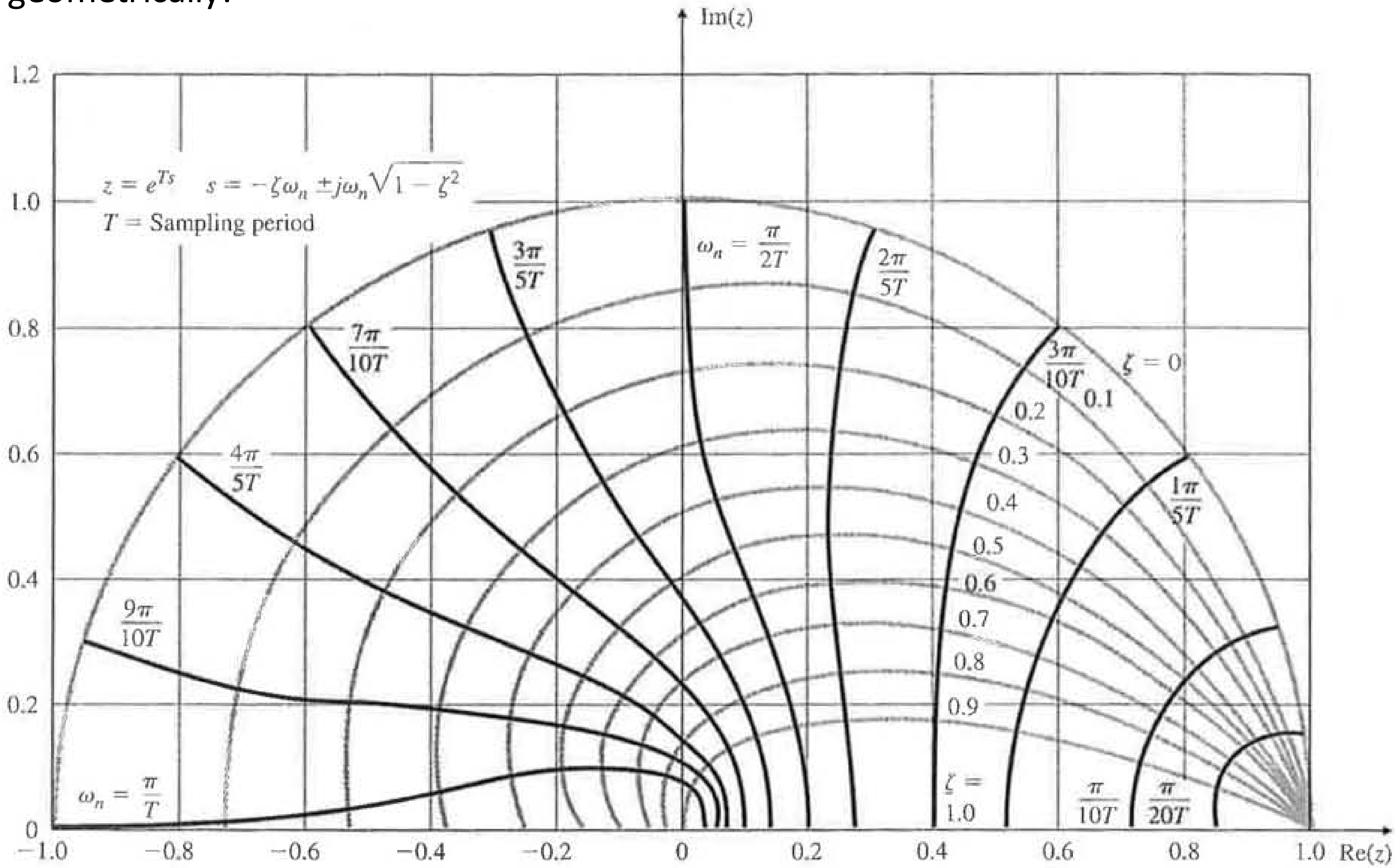


Figure 8.4

Natural frequency (solid color) and damping loci (light color) in the z-plane; the portion below the Re(z)-axis (not shown) is the mirror image of the upper half shown

$$\zeta = \frac{-\ln r}{\sqrt{\ln^2 r + \theta^2}} \qquad \omega_n = \frac{1}{T}\sqrt{\ln^2 r + \theta^2}$$

These relationships between the locations of a pole pair at (r, ±θ) in the z-plane and second order system parameters (ζ, ω_n) allow us then to relate pole locations to “boss parameters” such as %OS and settling time.

Example:

We have a DT system with the following closed-loop characteristic polynomial:

$$z^2 - z + 0.632 = (z - 0.5 - j(0.618)(z - 0.5 + j0.618) = 0$$

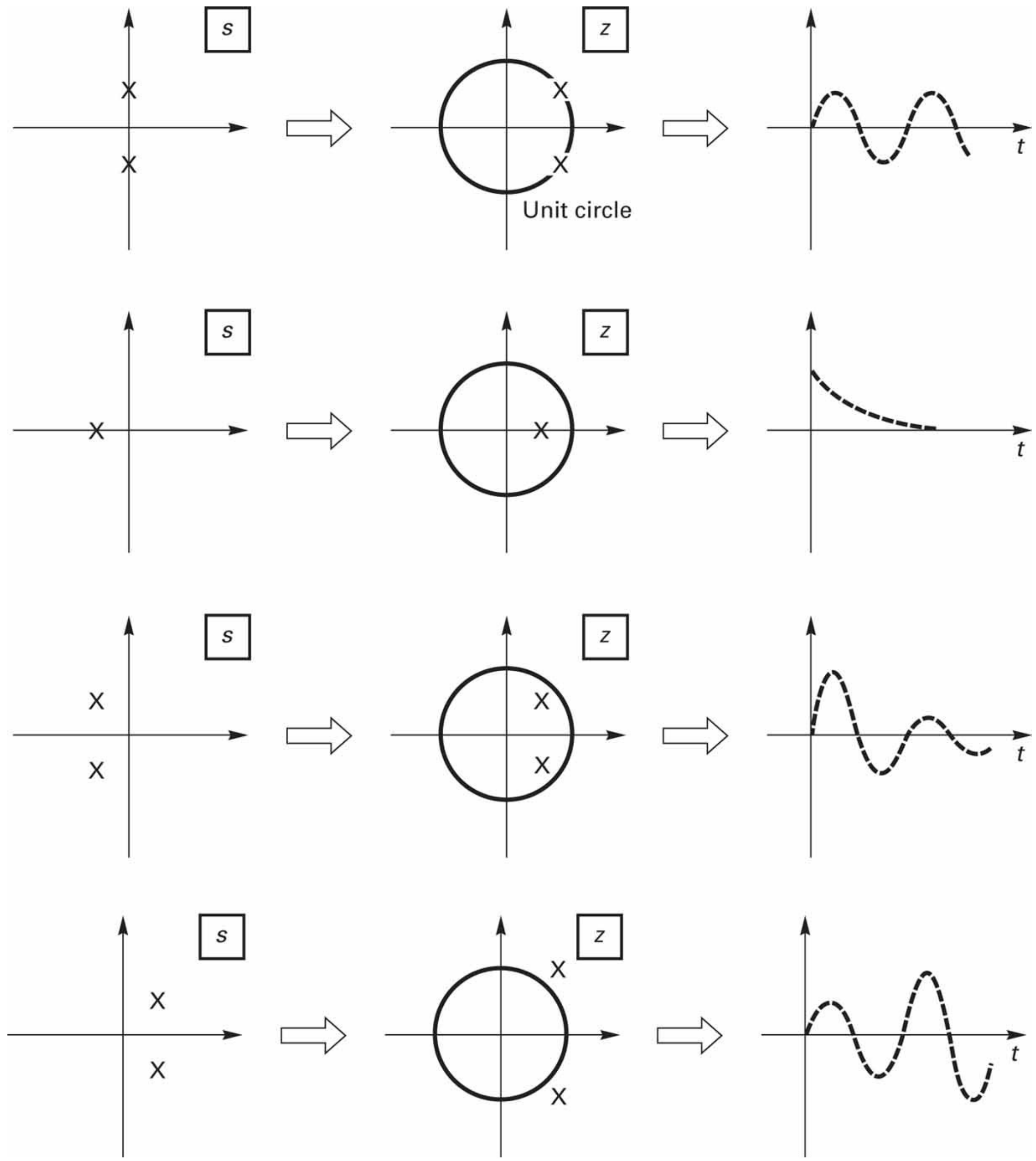
Get the pole locations in the z-plane in terms of (r, ±θ) then obtain the 2nd order parameters (in this example T = 1s which is rather slow):

$$z_{1,2} = 0.5 \pm j0.618 = 0.795\angle \pm 0.890(\text{rad}) = r\angle\pm\theta$$

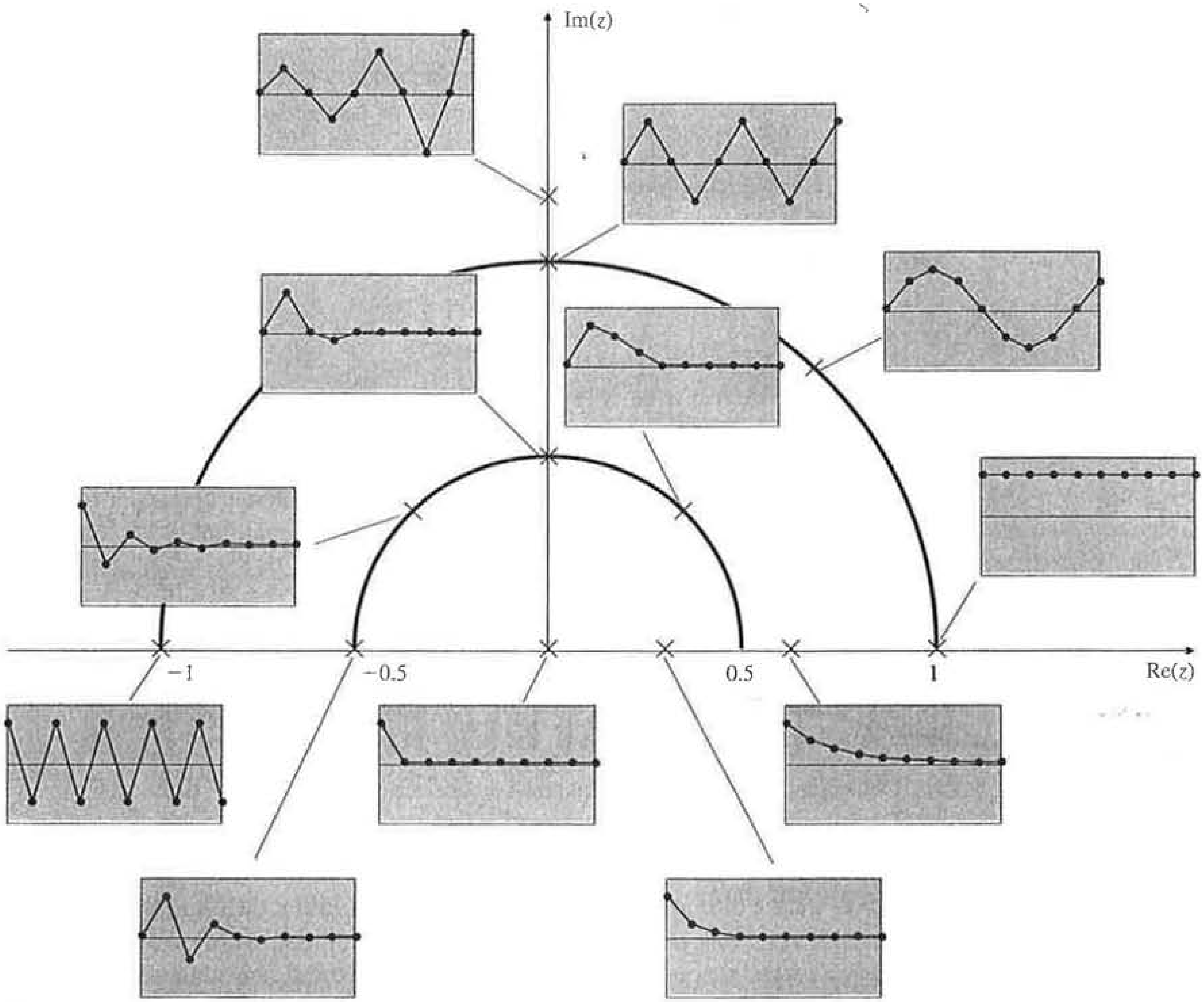
$$\zeta = \frac{-\ln(0.795)}{\sqrt{\ln^2(0.795) + (0.890)^2}} = 0.250$$

$$\omega_n = \frac{1}{1}\sqrt{\ln^2(0.795) + (0.890)^2} = 0.919$$

The examples below illustrate 4 different configurations of s-plane and corresponding z-plane pole locations and the resulting signals produced.

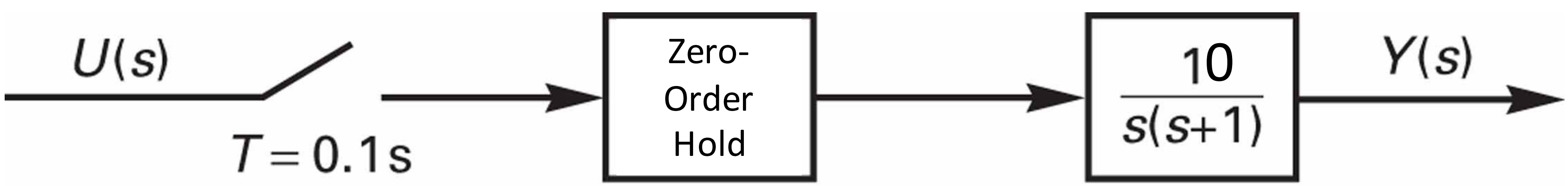


The following plot from Franklin gives a similar picture:



Digital State Feedback Design

- State feedback can be applied to sampled data systems in almost exactly the same way as for CT systems
  - The only real difference is that we place eigenvalues in the z-plane, not the s-plane
- We proceed by example. Assume we have the following servomotor system (again):



In the previous set of notes we developed the following discretized state-space model for this system:

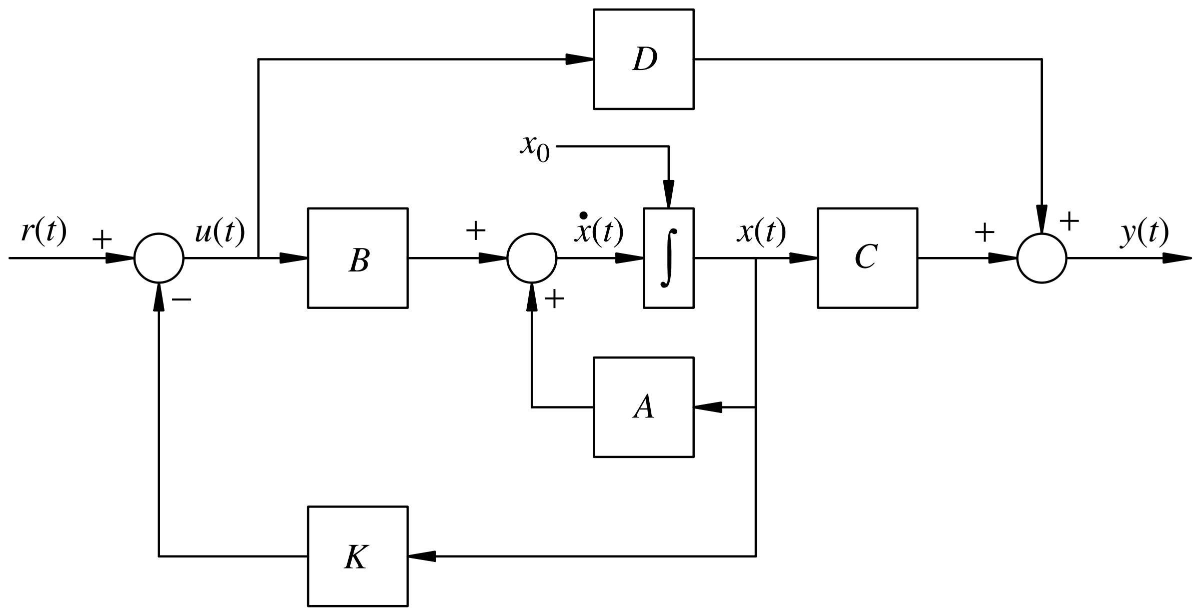
$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0.0484 \\ 0.952 \end{bmatrix} m(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}(k)$$

$x_1(k)$ represents the angle of the motor shaft (measureable by encoder count).
$x_2(k)$ represents the shaft speed (measureable by a tachometer, rate gyro, or by rate of encoder counts).

It is important to consider whether the state variables are measureable because otherwise full-state feedback cannot be applied.

Here is our usual picture of a state feedback controller:



This example differs in that it has been discretized, but also in that the goal is to set the motor's shaft angle to zero. That makes this controller a **regulator**. A regulator is a controller or compensator that works to move one or all state variables to zero. So we can say there is no r(t), or equivalently that r(t) = 0.

In regulator design (for n=2) the input to the plant is defined as

$$u(k) = -K_1x_1(k) - K_2x_2(k) = -\mathbf{Kx}(k)$$

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.0484 \\ 0.952 \end{bmatrix} m(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

Problem specification: Reduce settling time to 4 seconds. (Nothing else is mentioned which means we don't particularly care about other specifications such as %OS).

Start by looking at the open-loop system and its characteristics. We will need the current characteristic polynomial (computed as usual except that we use |zI – A| instead of |sI – A|).

$$a(z) = |zI - A| = (z-1)(z-0.905) = z^2 - 1.905z + 0.905$$

The design process that follows goes from a unity feedback system (which is identical to state feedback with K_1 = 1, K_2 = 0). That unity feedback system has the following characteristic polynomial:

$$a_{uf}(z) = z^2 - 1.9z + 0.91$$

The eigenvalues of the unity feedback system can be obtained from the quadratic formula then converted to polar form:

$$z_{1,2} = 0.954 \underline{/\pm 0.091\,\mathrm{rad}} = r\underline{/\pm\theta}$$

$$z_{1,2} = 0.954 \angle \pm 0.091 \text{ rad} = r \angle \pm \theta$$

Work out the second-order parameters:

$$\zeta = \frac{-\ln r}{\sqrt{\ln^2 r + \theta^2}} = \frac{-\ln(0.954)}{\sqrt{\ln^2(0.954) + (0.091)^2}} = 0.46$$

$$\omega_n = \frac{1}{T}\sqrt{\ln^2 r + \theta^2} = 1.0246$$

Current settling time:

$$T_s = \frac{4}{\zeta\omega_n} = 8.47$$

Since we don't care about %OS lets just change ω_n. To bring the desired settling time down to 4 seconds we modify ω_n and then get the desired pole locations:

$$\omega_n' = \frac{4}{\zeta 4} = 2.17$$

$$z_{1,2} = e^{sT}\Big|_{s=s_{1,2}} = e^{-\zeta\omega_n T} e^{\pm j\omega_n T\sqrt{1-\zeta^2}} = re^{\pm j\theta}$$

$$\lambda_{1,2} = 0.905 \angle \pm 11.04° = 0.888 \pm j0.173$$

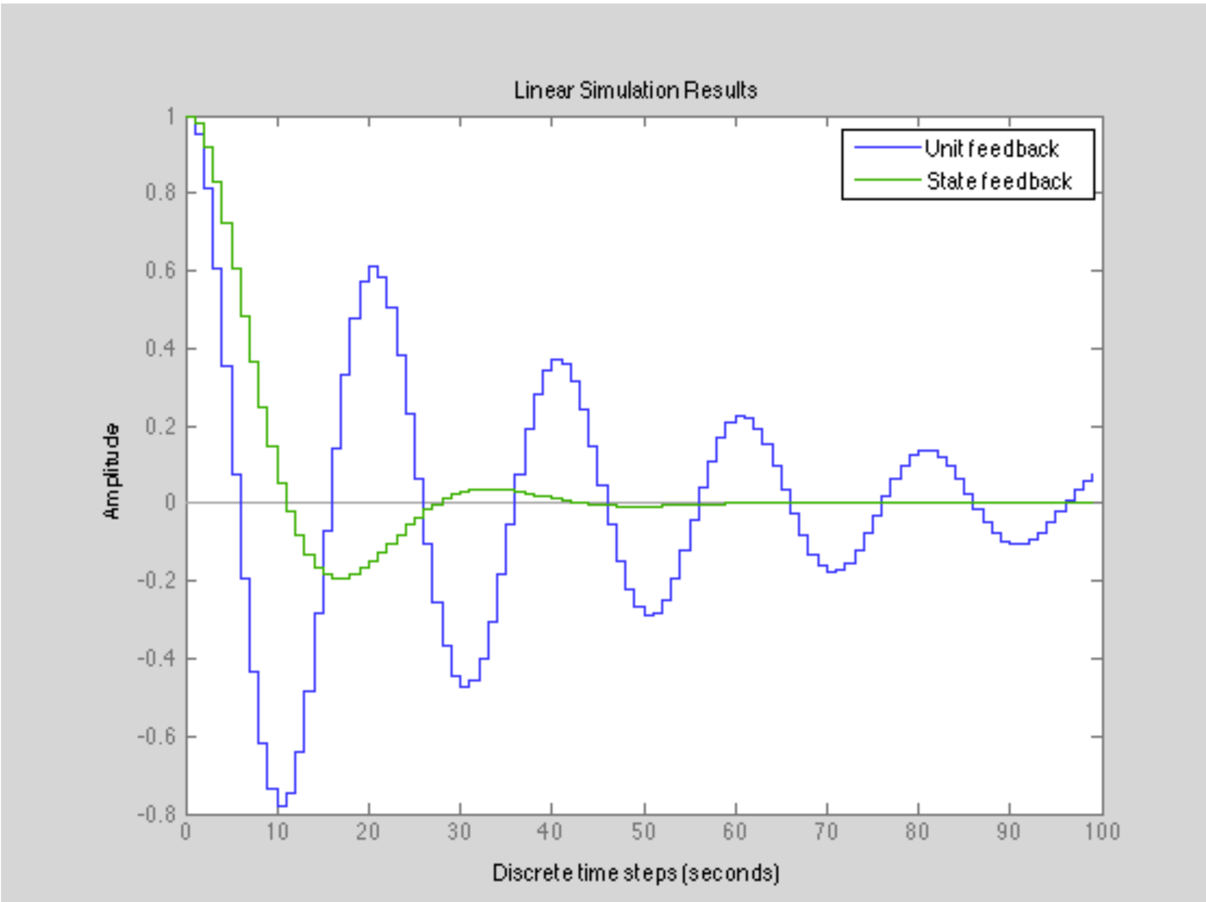
$$\lambda_{1,2} = 0.905\angle{\pm 11.04^\circ} = 0.888 \pm j0.173$$

Now we can get the desired characteristic polynomial:

$$(z - 0.888 - j0.173)(z - 0.888 + j0.173) = z^2 - 1.776z + 0.819$$

We continue to design the K gain vector in the usual way. The system is not in CCF so we use Bass-Gura and obtain K = [0.445   0.113].

The following shows the resulting improvement in system response ($x(0) = [1 \;\; 0]^T$).



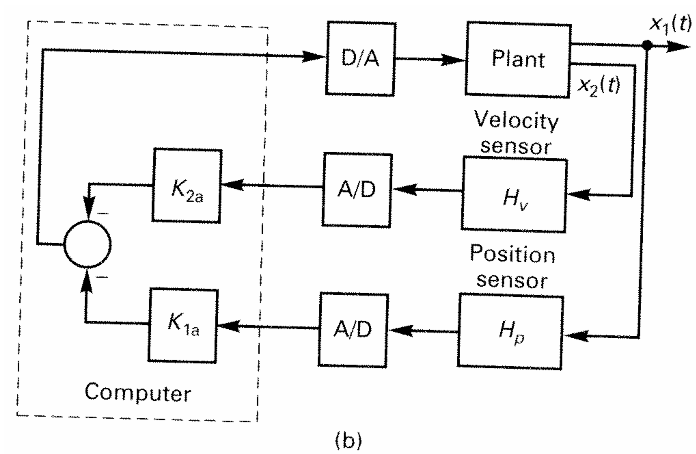


FIGURE 14.3
Hardware implementation for the design of Example 14.1.