ENGI 7825: Linear Algebra Review
Linear Combinations, Span, and Independence

Adapted from Notes Developed by Martin Scharlemann

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Definition

Suppose $\{c_1, c_2 \ldots c_k\}$ are all real numbers.
The vector
$$\vec{y} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$
is called a linear combination of the vectors $\{\vec{v}_1, \vec{v}_2 \ldots \vec{v}_k\}$.

Sample problem:

Given vectors $\{\vec{a}_1, \vec{a}_2 \ldots \vec{a}_n, \vec{b}\}$ in $\mathbb{R}^m$, find real numbers $\{c_1, c_2 \ldots c_n\}$ so that
$$c_1\vec{a}_1 + \cdots + c_n\vec{a}_n = \vec{b}.$$

$$c_1\vec{a}_1 + \cdots + c_n\vec{a}_n = \vec{b}:$$

Solving this system requires solving these m linear equations:

$$\begin{aligned} a_{11}c_1 + \ldots + a_{1n}c_n &= b_1 \\ a_{21}c_1 + \ldots + a_{2n}c_n &= b_2 \\ &\vdots \\ a_{m1}c_1 + \ldots + a_{mn}c_n &= b_m \end{aligned}$$

The system has augmented matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right]$$

Our goal now is to solve this system by transforming this augmented matrix into reduced echelon form (zeroes above pivots).

Example:

Suppose

$$\vec{a}_1 = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 8 \end{bmatrix} \quad \vec{a}_2 = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 8 \end{bmatrix} \quad \vec{a}_3 = \begin{bmatrix} 6 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \vec{a}_4 = \begin{bmatrix} 0 \\ 6 \\ 10 \\ 26 \end{bmatrix}$$

and want to find c_1, c_2, c_3, c_4 so that

$$c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 + c_4\vec{a}_4 = \begin{bmatrix} 12 \\ 4 \\ 13 \\ 23 \end{bmatrix}$$

This translates to the system of linear equations whose augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 0 & 6 & 0 & 12 \\ 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ 8 & 8 & -1 & 26 & 23 \end{array}\right]$$

applying Gauss-Jordan elimination leads to:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and so has general solution

$$c_2 = \textit{anything}, \; c_1 = \frac{3}{2} - c_2, c_3 = 2, c_4 = \frac{1}{2}$$

Definition

Given a collection $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ of vectors in $\mathbb{R}^m$, the set of all linear combinations of these vectors, that is all vectors that can be written as

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$

for some $c_1, \ldots, c_k \in \mathbb{R}$ is denoted

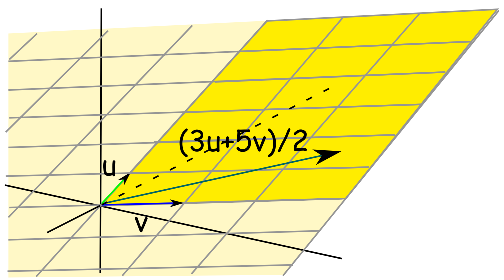
$$\text{Span}\,\{\vec{v}_1, \ldots, \vec{v}_k\}$$

and is called the span of $\{\vec{v}_1, \ldots, \vec{v}_k\}$.

Easy example: If $k = 1$ so there is only one vector $\vec{v}$, then $\text{Span}\{\vec{v}\}$ is just all vectors that are multiples of $\vec{v}$. That is, $\text{Span}\{\vec{v}\} = \{c\vec{v} \mid c \in \mathbb{R}\}$

When there is only one vector $\vec{v}$ then $\text{Span}\{\vec{v}\} = \{c\vec{v} \mid c \in \mathbb{R}\}$ is just the line that contains both $\vec{0}$ (take $c = 0$) and $\vec{v}$ (take $c = 1$).

With two vectors $\vec{u}$ and $\vec{v}$, $\text{Span}\{\vec{u}, \vec{v}\} = \{c_1\vec{u} + c_2\vec{v}\}$ pictured via the parallelogram rule (Span = entire plane; $c_i \geq 0$ highlighted):



Repeat of example above: The matrix and reduced echelon form:

$$\begin{bmatrix} 0 & 0 & 6 & 0 & 12 \\ 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ 8 & 8 & -1 & 26 & 23 \end{bmatrix} \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

has general solution:

$$c_2 = \textit{anything}, \; c_1 = \frac{3}{2} - c_2, c_3 = 2, c_4 = \frac{1}{2}$$

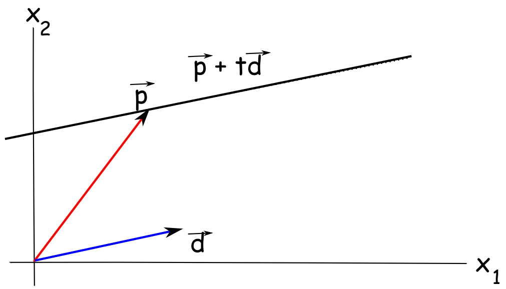
We can write this as

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ 2 \\ \frac{1}{2} \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

So we can think of the set of all solutions as

$$\begin{bmatrix} \frac{3}{2} \\ 0 \\ 2 \\ \frac{1}{2} \end{bmatrix} + \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

So we can picture the solution as a line in the direction of the second vector, going through the point given by the first vector (but in $\mathbb{R}^4$!)



Example with two free variables $\{x_2, x_4\}$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & \pi & 0 & \frac{3}{2} \\ 0 & 0 & 1 & e & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = \frac{3}{2} - x_2 - \pi x_4 \\ x_3 = 2 \qquad\quad - e x_4 \\ x_5 = \frac{1}{2} \end{array}$$

which can be written:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ 2 \\ 0 \\ \frac{1}{2} \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\pi \\ 0 \\ -e \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ 2 \\ 0 \\ \frac{1}{2} \end{bmatrix} + \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\pi \\ 0 \\ -e \\ 1 \\ 0 \end{bmatrix} \right\}$$

Background thoughts:

If two non-trivial vectors $\vec{x}_1, \vec{x}_2$ both lie on the same line, then $Span\{\vec{x}_1, \vec{x}_2\}$ is just that line.

On the other hand, if they *don't* lie on the same line, then $Span\{\vec{x}_1, \vec{x}_2\}$ consists of an entire plane.

So the span of two vectors may be a plane, or it could be something simpler: either a line, or even just $\vec{0}$ in the case that $\vec{x}_1 = \vec{0} = \vec{x}_2$.

Similarly, if *three* non-trivial vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3 \in \mathbb{R}^3$ all lie on the same line, then $Span\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is just that line.

If they don't all lie on the same line, but lie on the same plane, then $Span\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is just that plane.

If they don't all lie in the same plane, then $Span\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is $\mathbb{R}^3$.

Definition

A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ in $\mathbb{R}^m$ is linearly independent if and only if the only solution to the equation

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$$

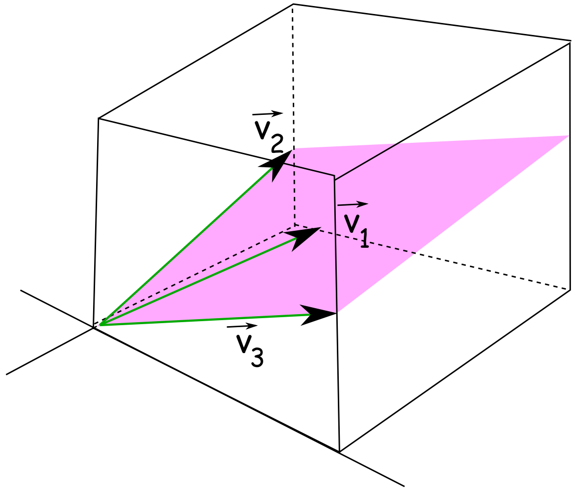
is the solution $c_i = 0$ for $1 \le i \le k$.
Conversely, the set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent if there are real numbers $c_1, ..., c_k$, not all zero, such that

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}.$$

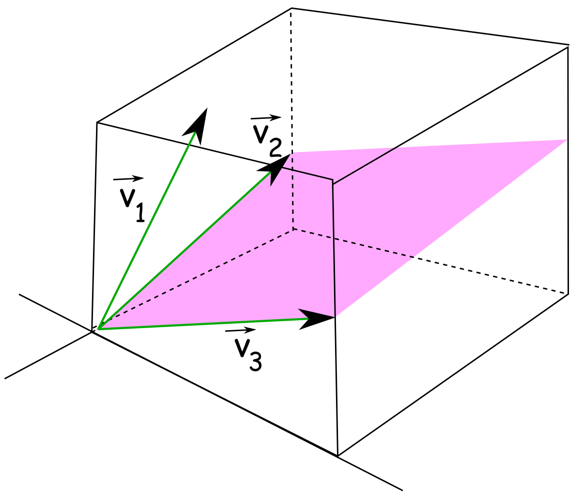
Idea: if the set is linearly independent, then the span is as big as possible.

If the set is linearly dependent then the span is somehow "thinner"; you could even remove some vectors and not change the span.

In pictures:



$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly dependent.



$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly independent.

Property 1: If even a single $\vec{v}_i = \vec{0}$ then $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent. Why?

Suppose that $\vec{v}_1 = \vec{0}$. Then

$$1\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 \cdots + 0\vec{v}_k = \vec{0}$$

yet $c_1 = 1 \neq 0$.

Property 2: If even a single $\vec{v}_i$ is a multiple of a different $\vec{v}_j$ then $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent. Example: Let $\vec{v}_2 = 5\vec{v}_1$. Then

$$5\vec{v}_1 - \vec{v}_2 + 0\vec{v}_3 \cdots + 0\vec{v}_k = \vec{0}$$

yet $c_1 = 5 \neq 0$ (and also $c_2 = -1 \neq 0$).

Property 3: If any subset of $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent, so is the whole set.

Question: Is this set of vectors linearly dependent, or linearly independent?

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \right\}$$

A) Dependent since they are $2d$ vectors in $\mathbb{R}^3$ and $2 < 3$.
B) Independent since they are $2d$ vectors in $\mathbb{R}^3$ and $2 < 3$.
C) Dependent because one is a multiple of the other.
D) Independent because neither is a multiple of the other.
E) Independent because one is a multiple of the other.

Answer: D

Question: Is this set of vectors linearly dependent, or linearly independent? (There are two correct answers.)

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -13 \end{bmatrix} \right\}$$

A) Independent since they are 3 vectors in $\mathbb{R}^3$.
B) Dependent because one is a multiple of the other.
C) Dependent because a subset is dependent.
D) Independent because a subset is independent.

Answer: B and C