ENGI 7825: Control Systems II

The State-Space Representation: Part 4: Linearization

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Adapted from the notes of Gabriel Oliver Codina
A function, $f$, is linear if it exhibits the following properties:

- **Additivity**: $f(p + q) = f(p) + f(q)$
  - Also known as the superposition principle
- **Homogeneity**: $f(\alpha p) = \alpha f(p)$

Consider the function $f(x) = mx + b$:

- **Additivity?**
  - $f(p) = mp + b$
  - $f(q) = mq + b$
  - $f(p + q) = m(p + q) + b = mp + mq + b$
  - This is not equal to $f(p) + f(q)$!
  - So the equation for a straight line is not linear!!
  
Although it would be linear if $b = 0$. 
Linear Differential Equations

► For a differential equation to be linear, it must be possible to form linear combinations of solutions which are also solutions. This is the case for any DE of the following form:

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)}(t) + \cdots + a_0 y(t) = g(t) \]

► This is a non-homogenous (RHS is not 0) DE with constant coefficients.

► The SS representation can represent multiple linear DE’s together.

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t) \\
x(t_0) &= x_0
\end{align*}
\]

► Our ability to describe the system in SS form implies a linear system.
Linearization of nonlinear systems

- The SS representation for a nonlinear time-varying system is as follows:

\[
\begin{align*}
\dot{x}(t) &= f[x(t), u(t), t] \\
y(t) &= h[x(t), u(t), t]
\end{align*}
\]

\[x(t_0) = x_0\]

- A system is nonlinear if it cannot be written in the standard SS form.

- There exist techniques to solve some nonlinear state equations, but they will not be exposed in this course.

- We will use a first-order Taylor series expansion to linearize such a system about a particular operating point.
The Taylor series expansion gives the value of a function, $f$, at $t$ from its value at $a$ and its derivatives evaluated at $a$:

$$f(t) = f(a) + \frac{f'(a)}{1!}(t - a) + \frac{f''(a)}{2!}(t - a)^2 + \frac{f'''(a)}{3!}(t - a)^3 + \cdots$$

The higher-order terms are typically small, so we often make a first-order approximation by eliminating them:

$$f(t) \approx f(a) + \frac{f'(a)}{1!}(t - a)$$

The point $a$ is called the operating point and $f(a)$ is the nominal value.
1D Example

► Find a linear approximation for the function $f(t) = t^2$ at the operating point $t = 1$
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► The resulting function, $2t - 1$, is not linear!

► However, we can define a new deviation variable $f_\delta(t)$ with respect to the nominal value of the function at $t = 1$

$$f_\delta(t) = f(t) - f_n(t)$$
$$= (2t - 1) - (1)$$
$$= 2t - 2$$

► This is still not linear, but we can make it linear by defining a new time variable, $t_\delta$, defined as $t_\delta = t - 1$. When we substitute for $t = t_\delta + 1$ we get,

$$f_\delta(t_\delta) = 2(t_\delta + 1) - 2 = 2t_\delta$$
2D Example

- Linear approximation for a two variable function $f(x,y)$
  - A linear approximation for $f(x,y)$ about $(a,b)$, is obtained with a first-order multivariate Taylor series expansion

$$f(x,y) \approx f(a,b) + \left[ \frac{\partial f}{\partial x} \right]_{(a,b)}(x-a) + \left[ \frac{\partial f}{\partial y} \right]_{(a,b)}(y-b)$$

Example: Find the linearization of $f(x,y)=x^2+y^2$ about point $(1,2)$

$$f(1,2) = 5$$

$$\left[ \frac{\partial f}{\partial x} \right]_{(1,2)} = 2x\big|_{(1,2)} = 2$$

$$\left[ \frac{\partial f}{\partial y} \right]_{(1,2)} = 2y\big|_{(1,2)} = 4$$

$$f(x,y) = 2x + 4y - 5$$
Linearization of nonlinear systems

Nonlinear, time-varying systems can be represented by state equations:

\[ \dot{x}(t) = f[x(t),u(t),t] \]
\[ y(t) = h[x(t),u(t),t] \]
\[ x(t_0) = x_0 \]

where \( f \) and \( h \) are continuously differentiable functions. Linearization is obtained as follows:

- Assume that under usual working circumstances the system operates along the trajectory \( x_n(t) \) while it is driven by the system input \( u_n(t) \) called the nominal state trajectory and the nominal input trajectory, respectively.

\[ \dot{x}_n(t) = f[x_n(t),u_n(t),t] \]
\[ y_n(t) = h[x_n(t),u_n(t),t] \]

- Expanding the nonlinear functions in a multivariate first-order Taylor series expansion about \([x(t), u(t), t]\) we obtain:

\[ \dot{x}(t) \approx f[x_n(t),u_n(t),t] + \frac{\partial f}{\partial x} [x_n(t),u_n(t),t] [x(t) - x_n(t)] + \frac{\partial f}{\partial u} [x_n(t),u_n(t),t] [u(t) - u_n(t)] \]
\[ y(t) \approx h[x_n(t),u_n(t),t] + \frac{\partial h}{\partial x} [x_n(t),u_n(t),t] [x(t) - x_n(t)] + \frac{\partial h}{\partial u} [x_n(t),u_n(t),t] [u(t) - u_n(t)] \]
This is exactly the same as our 2-variable example, except the variables that we differentiate by are now vectors, and the corresponding derivatives are matrices!

Let's say we have a function $f(\mathbf{v})$ where $\mathbf{v}$ is a vector. The derivative of $f$ is called the **Jacobian matrix** and is defined as follows if the output of $f$ is 2-dimensional and $\mathbf{v}$ is 3-dimensional.

\[
\mathbf{J} = \frac{\partial f}{\partial \mathbf{v}} = \begin{bmatrix}
\frac{\partial f_1}{\partial v_1} & \frac{\partial f_1}{\partial v_2} & \frac{\partial f_1}{\partial v_3} \\
\frac{\partial f_2}{\partial v_1} & \frac{\partial f_2}{\partial v_2} & \frac{\partial f_2}{\partial v_3} \\
\frac{\partial f_3}{\partial v_1} & \frac{\partial f_3}{\partial v_2} & \frac{\partial f_3}{\partial v_3}
\end{bmatrix}
\]

In general, let's say that $f$ is $m$-dimensional and $\mathbf{v}$ is $n$-dimensional.

\[
\frac{\partial f}{\partial \mathbf{v}} = \begin{bmatrix}
\frac{\partial f_1}{\partial v_1} & \cdots & \frac{\partial f_1}{\partial v_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial v_1} & \cdots & \frac{\partial f_m}{\partial v_n}
\end{bmatrix}
\]
We can define the following deviation variables:

\[ x_\delta(t) = x(t) - x_n(t) \quad u_\delta(t) = u(t) - u_n(t) \quad y_\delta(t) = y(t) - y_n(t) \]

Now we rearrange our big equation to use these:

\[
\begin{align*}
\dot{x}(t) &= f[x_n(t), u_n(t), t] + \frac{\partial f}{\partial x}[x_n(t), u_n(t), t][x(t) - x_n(t)] + \frac{\partial f}{\partial u}[x_n(t), u_n(t), t][u(t) - u_n(t)] \\
y(t) &= h[x_n(t), u_n(t), t] + \frac{\partial h}{\partial x}[x_n(t), u_n(t), t][x(t) - x_n(t)] + \frac{\partial h}{\partial u}[x_n(t), u_n(t), t][u(t) - u_n(t)]
\end{align*}
\]

\[
\begin{align*}
\dot{x}(t) - \dot{x}_n(t) &= \frac{\partial f}{\partial x}[x(t), u_n(t), t][x(t) - x_n(t)] + \frac{\partial f}{\partial u}[x_n(t), u_n(t), t][u(t) - u_n(t)] \\
y(t) - y_n(t) &= \frac{\partial h}{\partial x}[x_n(t), u_n(t), t][x(t) - x_n(t)] + \frac{\partial h}{\partial u}[x_n(t), u_n(t), t][u(t) - u_n(t)]
\end{align*}
\]

\[
\begin{align*}
\dot{x}_\delta(t) &= \dot{x}(t) - \dot{x}_n(t) \approx A(t)x_\delta(t) + B(t)u_\delta(t) \\
y_\delta(t) &= y(t) - y_n(t) \approx C(t)x_\delta(t) + D(t)u_\delta(t)
\end{align*}
\]
In many cases the non-linear functions, $f$ and $h$, will be time-invariant.

In this case, the matrices $A$, $B$, $C$, and $D$ will be constant and the linearized system will be LTI.

Another potential simplification occurs if the nominal trajectory just represents a constant equilibrium condition $x_n(t) = x_n$ for a constant nominal input $u_n(t) = u_n$. In this case, the derivative is zero:

$$0 = f[x_n, u_n]$$
The motion of a pendulum on a taut string of length $L$ is described by the following:

$$mL^2\ddot{\theta}(t) + mgL \sin(\theta(t)) = T(t)$$

The use of $\sin(\theta)$ makes this equation non-linear. We will linearize, but we first need to define our state variables. Assuming that $\theta(t)$ is the output:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$$
\[
\begin{bmatrix}
    x_1(t) \\
    x_2(t)
\end{bmatrix} = \begin{bmatrix}
    \theta(t) \\
    \dot{\theta}(t)
\end{bmatrix}
\]

\[mL^2\ddot{\theta}(t) + mgL \sin(\theta(t)) = T(t)\]

► Let's try and write in SS form and see how far we get:

\[\dot{x}_1(t) = x_2(t)\]

\[\dot{x}_2(t) = -\frac{g}{L} \sin(\theta(t)) + \frac{1}{mL^2}T(t)\]

► Non-linear! But let's keep going with the output equation:

\[y(t) = \theta(t) = [1 \ 0] \begin{bmatrix}
    x_1(t) \\
    x_2(t)
\end{bmatrix}\]

► The output equation is linear, so we only need to linearize the state equation
We will choose to linearize about a nominal input of $u_n(t) = 0$ and $x_n(t) = 0$. So our approximation will be good only around the stable equilibrium position (i.e. when $\theta$ is small). Here is the first-order Taylor expansion again, defined w.r.t. $x_n(t)$:

$$
\dot{x}(t) - \hat{x}_n(t) = \frac{\partial f}{\partial x}[x_n(t), u_n(t), t][x(t) - x_n(t)] + \frac{\partial f}{\partial u}[x_n(t), u_n(t), t][u(t) - u_n(t)]
$$

Since the nominal input and value are zero, we have:

$$
\dot{x}(t) = \frac{\partial f}{\partial x}[0,0,t][x(t)] + \frac{\partial f}{\partial u}[0,0,t][u(t)]
$$

$$
\frac{\partial f}{\partial x}(0,0,t) = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-\frac{g}{L} & 0
\end{bmatrix} \quad \frac{\partial f}{\partial u}(0,0,t) = \begin{bmatrix}
\frac{\partial f_1}{\partial u} \\
\frac{\partial f_2}{\partial u}
\end{bmatrix} = \begin{bmatrix}
0 \\
\frac{1}{mL^2}
\end{bmatrix}
$$

Finally, the linearized state equation:

$$
\dot{x}(t) = \begin{bmatrix}
0 & 1 \\
-\frac{g}{L} & 0
\end{bmatrix} x(t) + \begin{bmatrix}
0 \\
\frac{1}{mL^2}
\end{bmatrix} u(t)
$$